PREPRINT

Cascades of Period Doubling Bifurcations in n Dimensions

Eleonora Catsigeras *

Published in *Non Linearity* Num. 9, pp 1061-1070, 1996

Abstract

In the first part of this paper sufficient conditions are stated for a cascade of period doubling bifurcations in n dimensions be reducible to a perturbation of a map in the interval with critical points. In the second part we show that the example of a cascade in the n-dimensional disk, given by Gambaudo and Tresser, can be approximated by maps exhibiting homoclinic tangencies.

1 Introduction

A global program for locally dissipative dynamical systems has been formulated by J.Palis [P,1991] in which the idea is to define a dense subset $\mathcal{H} \subset \text{Diff}^3(M)$ (M is a compact manifold without boundary) and describe the prevalent dynamical phenomena in small neighborhoods of the elements of $\mathcal{H}$ in most k-parameters families of diffeomorphisms through them. If dim $M = 2$ a candidate for $\mathcal{H}$ is the union of the hyperbolic diffeomorphisms and the ones exhibiting homoclinic tangencies. The inspiring reason for taking the diffeomorphisms with homoclinic tangencies is the richness of the dynamics when unfolding the tangency along a one-parameter family: for instance cascades of period doubling bifurcations [YA,1983], diffeomorphisms exhibiting infinitely many coexisting sinks [N,1979], Hénon-like attractors [MV,1993].

To develop this program, the globally unstable systems, in particular maps where a cascade of period doubling bifurcations accumulate (that we call briefly a "cascade") and maps exhibiting a Feigenbaum's attractor, should be approximated with diffeomorphisms exhibiting homoclinic tangencies.

The question we are dealing with is whether we can perturb a cascade to obtain homoclinic tangencies. Mainly two types of n-dimensional cascades are considered: those that are intrinsically one-dimensional (that is, after sufficiently renormalized, it remains in a neighborhood of a map

*IMPA/CNPq Brasil and IMERL Fac. Ingenieria Uruguay. Address: IMERL Fac. Ingenieria C.C 30 Montevideo Uruguay. E-mail: eleonora@fing.edu.uy

in the interval) and those such that, although renormalized many times do not loose their n-dimensional character.

In section 2 of this work we give sufficient conditions for the cascade be of the first type. Although there is not a general result of approximation of this type of cascades with homoclinic tangencies, a positive answer is given in [C,1995] when the map of the interval is the Feigenbaum map, working in the analytic topology. In a work in progress we are extending this result to the C^r topology, for r sufficiently large, depending on the dimension, but still in a neighborhood of the Feigenbaum map. Possibly this is also true for perturbations of unimodal maps of combinatorial type 2^∞, with nondegenerate critical point (i.e. with nonvoid quadratic term) due to the fact that the renormalizations of these maps converge to the Feigenbaum map [S,1991].

In [GT,1992] it is constructed a n-dimensional cascade of the second type mentioned above (not reducible to a map in the interval), fixed by the renormalization, showing that the Feigenbaum's universality does not hold in a n-dimensional setting. A positive answer to the question of approximation with homoclinic tangencies is given for this case, proved in the section 3 of this work, exploring the geometric method of the construction of the cascade.

In an analytic neighborhood of the Feigenbaum map [F,1978] [CT,1978] the one-parameter families of one-dimensional maps that are transversal to the stable manifold of the renormalization transformation, exhibit cascades of period doubling bifurcations of sinks [CEK,1981], in which the sink is transformed into a saddle of stable codimension one and generates a new sink of double period. We will consider the map f at the accumulation of the bifurcations. It exhibits periodic saddles of period 2^n for each $n \geq 0$, and a Cantor set attracting all the orbits in the complement of the stable manifolds of the saddles. Successive renormalizations of the map f converge to the Feigenbaum one-dimensional map. The geometry of the Cantor set is then bounded: it has almost affine copies of itself (the changes of variables needed to renormalize the family are near affine transformations). As the family exhibiting the cascade is close to a one-dimensional family, the map f is globally dissipative, that is the determinant of the Jacobian matrix is smaller than one. We show that, in dimension two, the boundness of the geometry and the dissipativeness are enough for a general cascade of class $C^r(r \geq 3)$ to be reducible to dimension one.

We start from a general cascade f in the n-dimensional ball of class $C^r, r \geq 1$. First we reduce to dimension two, taking the quotient map on the leaves of a contractive invariant foliation. Next, for $r \geq 3$, under the assumptions of uniform dissipativeness and boundness of the geometry, we obtain that high iterates of the cascade in dimension two approximate a map in the interval having at least one critical point and periodic orbits of periods the powers of two.

As mentioned before the one-parameter unfolding of a homoclinic tangency for locally dissipative maps of class $C^r(r \geq 3)$ in dimension two originates important globally unstable dynamical phenomena. Perhaps the simplest are the cascades of period doubling bifurcations of sinks, that appear because horseshoes are created when the tangency is unfolded [YA,1983]. The family exhibiting the cascade is near the one-dimensional quadratic family in two dimensions, after a proper renormalization of the cascade near the tangency [PT,1993]. The existence of cascades of period doubling is also valid in higher dimensions, when unfolding a homoclinic tangency [M,1991]. These examples of n-dimensional cascades are reducible to dimension one.

The author thanks J. Palis for posing the problem and for helpful discussions.

2 Reduction of the dimension

Let B_0 be a domain in R^n (i.e. a connected and bounded set that is the closure of its interior).

Definition 2.1 A *cascade of period doubling* is a map $f : B_0 \subset R^n \mapsto \text{int } B_0$ of class C^r, $r \geq 1$, provided with a family of domains $B_{m,j},\ m \geq 0,\ 0 \leq j \leq 2^m - 1,\ B_{0,0} = B_0$ such that

a) $B_{m,j} \cap B_{m,k} = \emptyset$ for $j \neq k$ and $B_{m,j} \subset \text{int } B_{m-1,j\ (mod\, 2^{m-1})}$ for $m \geq 1$.

b) diam $B_{m,j} \to 0$ with $m \to \infty$ uniformly in j.

c) $f(B_{m,j}) \subset B_{m,j+1\ (mod\, 2^m)}$ and $f^{2^m}(B_{m,j}) \subset \text{int } B_{m,j}$.

d) for each $m \geq 0$ there exists a periodic hyperbolic orbit of saddle type, of stable codimension one with negative expansing eigenvalue, of period 2^m with one point in $B_{m,j}$ for each $j = 0, 1, \ldots,\ 2^m - 1$, and there are no other periodic points.

e) for any $q \in B_0$ and all $m \geq 0$ the ω-limit of q is contained in the union of the periodic orbits of period $1, 2, \ldots, 2^{m-1}$ with $\cup_{j=0}^{2^m-1} \text{int} B_{m,j}$.

The definition above implies that the periodic orbit of period 2^m and its stable manifold are disjoint with $\cup_{j=0}^{2^{m+1}-1} B_{m+1,j}$.

Let p_m be the periodic point of period 2^m in $B_{m,0}$. The points of $B_{m,0} \backslash W^s(p_m)$ are classified in two sets: those points having an iterate by $f^{2^{m+1}}$ (and all of its following iterates) in $\text{int}(B_{m+1,0})$ and those having an iterate in $\text{int}(B_{m+1,2^m})$. They are open sets. Thus $W^s(p_m)$ disconnects the domain $B_{m,0}$.

We define $K = \cup_m \cap_j B_{m,j}$. Due to b) K is a Cantor set. Due to e) it is an attractor.

In [BGLT,1993] the wandering sets obtained without the convergence assumption (point b)of the definition above) are studied. In particular the authors find a C^1 example in which K contains a wandering domain. Also they prove that for $C^{1+\alpha}$ diffeomorphisms with a hypothesis of bounded geometry, the connected components of K have zero Lebesgue measure.

Remark 2.2 More generally, we will also call f a cascade of period doubling when f is not provided with the whole family of the domains $B_{m,j}$ of all the generations, but only with a subfamily $\{B_{m_k,j}, 0 \leq j \leq 2^{m_k} - 1\}, m_k \to \infty$ of the domains of generations m_k.

Each set $B_{m_k,j}$ must be provided with a saddle type point p_{m_k} of period 2^{m_k} and 2^i points of period 2^{m_k+i} for $0 < i < m_{k+1} - m_k$.

They must fulfill the properties a) to e) (with the obvious change of notation).

Our purpose is to reduce the dimension. We do this in two steps:

First, when there is a contractive invariant foliation of codimension 2, we can reduce f to a 2-dimensional cascade. This is proposition 2.4. Unfortunately there is not an invariant foliation of codimension one, when working with diffeomorphisms that exhibit a cascade of period doubling, as shown in proposition 2.3.

Second, we reduce to dimension one under some hypothesis of area contractiveness, and of uniform bounds of the sucessive renormalizations. This is done in theorem 2.7.

Proposition 2.3 *Let $f : B_0 \subset R^n \mapsto f(B_0) \subset B_0$ be a C^r diffeomorphism ($r \geq 1$) that is a cascade of period doubling. Then it does not exist a C^1 contractive foliation invariant by f of codimension one.*

Proof: By contradiction suppose that there exists an invariant C^1 foliation. Consider f^{2^m} with m sufficiently large to have $B_{m,0}$ contained in a neighborhood where the foliation can be trivialized. After the trivialization each of the leaves of the foliation is a subspace of codimension one, that separates R^n in two semispaces. In $B_{m,0}$ there is a fixed point p_m of f^{2^m} and a period 2 point p_{m+1}, both with negative expansive eigenvalues of $Df^{2^m}(p_m)$ and $Df^{2^{m+1}}(p_{m+1})$ respectively. Connect p_m and p_{m+1} with a continuous curve inside $B_{m,0}$. Applying $Df^{2^{m+1}}$ to any leaf of the foliation intersecting $B_{m,0}$ we obtain another leaf intersecting $B_{m,0}$. Call H to the set of points of the curve such that $Df^{2^{m+1}}$ maps each semispace onto itself. For instance p_m is in H. Call K to the set of points where the $Df^{2^{m+1}}$ maps each subspace onto the opposite. The point p_{m+1} is in K. As the map f is a diffeomorphism the sets H and K are open and complementary in the curve. This contradicts the connectedness of the curve. □

Proposition 2.4 (Reduction to dimension 2) *Let $f : B_0 \subset R^n \mapsto f(B_0) \subset B_0$ be a C^r diffeomorphism ($r \geq 1$) that is a cascade of period doubling.*

Suppose that there exists in B_0 a C^r foliation that is f-invariant and contractive by f, of codimension two.

Then there exist an integer m, a domain B_m invariant by f^{2^m}, and a C^r system of coordinates $\{(x,y) : x \in R^2, y \in R^{n-2}\}$ in B_m such that in those coordinates

$$f^{2^m}(x,y) = (g(x), h(x,y))$$

where g is a cascade of period doubling in dimension two.

Proof: Let us take $B_{m,j}, 0 \leq j \leq 2^m - 1$ the domains of generation m in the definition of cascade. They are invariant by f^{2^m}. Take m large enough such that $B_{m,0}$ is contained in the domain of a trivializing chart of the given foliation. In such trivializing coordinates $\{(x,y) : x \in R^2, y \in R^{n-2}\}$ the leaves are obtained fixing x. As they are invariant by f, we have:

$$f^{2^m}(x,y) = (g(x), h(x,y))$$

We call B_m to $B_{m,0}$.

We shall see that g is a cascade. Let us call Π to the projection on the first coordinate plane. (It is the projection along the leaves of the foliation). Define $\tilde{B}_0 = \Pi(B_m)$. It contains the point $x_0 = \Pi(p_m)$, where p_m is the periodic point of f in B_m with period 2^m. As $f^{2^m}(x_0, y_0) = (g(x_0), h(x_0, y_0)) = (x_0, y_0)$, we have that x_0 is a fixed point of g. It is hyperbolic of saddle type with stable codimension one and negative expansive eigenvalue: in fact the matrix $Df^{2^m}(x_0, y_0)$ is triangular and thus the eigenvalues of Df^{2^m} are those of Dg and those of $D_y h$. The last ones are in modulus smaller than one because the foliation is contractive. Then the expansive eigenvalue of Df^{2^m} is found in Dg.

The stable manifold of p_m contains all the leaves of the foliation that intersects, because the foliation is contractive with f, and thus for any q in the leaf through p:

$$\|f^{j2^m}(q) - f^{j2^m}(p)\| \to 0 \ \text{ with} j \to \infty$$

As $f^{j2^m}(p) \to p_m$ with $j \to \infty$, if $p \in W^s(p_m)$, it is obtained that $f^{j2^m}(q) \to p_m$. So $q \in W^s(p_m)$.

$\Pi(W_s(p_m))$ is the stable manifold of x_0, because if $f^{2^m j}(q) \to p_m$ then

$$g^j(\Pi(q)) = \Pi f^{2^m j}(q) \to \Pi(p_m) = x_0$$

The subdomains $\tilde{B}_{1,0}$ and $\tilde{B}_{1,1}$ for g are defined as the projections of the two subdomains of f of generation $m+1$ contained in B_m. They are disjoint: in fact, by contradiction suppose that one leaf of the foliation intersects the two subdomains of f of generation $m+1$ in points q_1 and q_2

respectively. As they are in the same contractive leaf, when iterating them by $f^{2^{m+1}}$ their distance converges to zero. But they are in different subdomains of generation $m+1$, that are invariant by $f^{2^{m+1}}$, closed and disjoint, thus having a positive distance.

All the points of $\tilde{B}_0 \setminus W^s(x_0)$ when sufficiently iterated by g land in $\tilde{B}_{1,0} \cup \tilde{B}_{1,1}$ because they are the projections of points of $B_m \setminus W^s(p_m)$. Thus there are not other fixed points of g besides x_0.

Analogously are constructed the periodic orbits of higher period for g and the domains of higher generation, projecting those of f. Finally the diameter of the domains of g are convergent to zero because they are the projection of the domains of f. $\square$

Definition 2.5 A C^r cascade of period doubling $f : B_0 \subset R^n \mapsto B_0$ is *renormalizable* if for all $m \geq 1$ the domains $B_{m,0}$ are C^r diffeomorphic to B_0.

Calling $\xi_f : B_0 \mapsto B_{1,0}$ to the diffeomorphism between the domains B_0 and $B_{1,0}$, we define *the first renormalized* of f as

$$Rf = \xi_f^{-1} \circ f \circ f \circ \xi_f : B_0 \mapsto B_0$$

Observe that Rf is also a renormalizable cascade of period doubling. The m renormalized of f is:

$$R^m f = \xi^{-1}_{R^{m-1}f} \circ R^{m-1} f \circ R^{m-1} f \circ \xi_{R^{m-1}}$$
$$= \xi^{-1}_{R^{m-1}f} \circ \ldots \circ \xi_f^{-1} \circ f^{2^m} \circ \xi_f \circ \ldots \circ \xi_{R^{m-1}f}$$

In particular cases we have some bounding properties of the change of variables $\xi_{R^m f}$. Frequently they are affine transformations.

Definition 2.6 We say that a C^r renormalizable cascade has *bounded geometry* if for all $m \geq 0$ the changes of variables $\xi_{R^m f}$ of the definition above are C^r bounded and there exist $\beta < 1$ and $\gamma > 0$, independent of m such that:

$$\max\{\|D\xi_{R^m f}\|_{C^0}, \|D(R^m f \circ \xi_{R^m f})\|_{C^0}\} \leq \beta < 1$$

and

$$|\det(D\xi_{R^m f})| \geq \gamma$$

Theorem 2.7 (Reduction of dimension 2 to dimension 1) *Let $f : B_0 \mapsto B_0 \subset R^2$ be a C^r $(r \geq 3)$ cascade of period doubling in two dimensions that is renormalizable and such that $R^m f$ is C^r bounded for all $m \geq 1$.*

Suppose that:

f is dissipative (i.e.: $0 \leq detDf(p) \leq \alpha < 1$ for all $p \in B_0$), and has bounded geometry.

Then:

There exists a C^{r-1} map g defined in B_0 such that:

For any given $\varepsilon > 0$ there exists an integer m verifying $\|g - R^m f\|_{C^{r-1}} \leq \varepsilon$,

There exist k, a domain $D \subset B_0$ invariant by g^{2^k}, and a C^{r-2} change of coordinates in D such that

$$g^{2^k}(x, y) = (g_1(x), g_2(x))$$

where g_1 is a multimodal map (i.e. with at least one critical point) of the interval.

Proof: Applying the Arzela-Ascoli theorem to the family of maps $R^m f$, there exists a subsequence m_j such that the $\lim_{j\to\infty} R^{m_j} f$ exists in the C^{r-1} topology. Let us call g to that limit.

We have:

$$\det Dg(q) = \lim_{j\to\infty} \det DR^{m_j} f(q)$$

$$\det DR^m f(q) = \prod_{i=0}^{m-1} \det D\xi_{R^i f}(q_{i+1}) \det Df^{2^m}(q_0) \prod_{i=0}^{m-1} \det(D\xi_{R^i f}(\tilde{q}_{i+1}))^{-1}$$

where $q_m = q,\ q_i = \xi_{R^i f} \circ \ldots \circ \xi_{R^{m-1} f}(q),\ i = 0, \ldots, m-1$ and $\tilde{q}_0 = f^{2^m}(q_0),\ \tilde{q}_{i+1} = \xi^{-1}_{R^i f} \circ \ldots \circ \xi_f(\tilde{q}_0),\ i = 0, \ldots, m-1$. Thus

$$|\det D(R^m f)(q)| \leq \alpha^{2^m} a^m \to 0 \text{ with } m \to \infty$$

where $a > 1$ is a uniform bound of the Jacobians of $\xi_{R^i f}$ and $\xi^{-1}_{R^i f}$.

Thus $\det Dg(q) = \lim_{j\to\infty} \det(DR^{m_j} f)(q) = 0$.

We already have a map g such that $\det Dg = 0$ in B_0, and $\|g - R^{m_j} f\|_{C^{r-1}} < \varepsilon$, for any given $\varepsilon > 0$, for all j sufficiently large, depending on ε.

As $\xi_{R^m f}$ and $R^m f$ are bounded uniformly in m in the C^r topology, there exist sucessive subsequences of $\{m_j\}$, making at each step i, $\xi_{R^{m_j+i} f}$ and R^{m_j+i} convergent in the C^{r-1} topology with i fixed and $j \to \infty$. Take the diagonal subsequence and then define, for each i:

$$R^i g = \lim_{j\to\infty} (R^{i+m_j}(f))$$

$$\bar{\xi}_i = \lim_{j\to\infty} \xi_{R^{m_j+i} f}$$

From

$$\xi_{R^{m_j} f} \circ \xi_{Rm_j+1f} \circ \ldots \xi_{R^{m_j+i-1} f} \circ R^{m_j+i} f = (R^{m_j} f)^{2^i} \circ \xi_{R^{m_j} f} \circ \xi_{Rm_j+1f} \circ \ldots \xi_{R^{m_j+i-1} f}$$

making $j \to \infty$ with i fixed, we obtain:

$$\bar{\xi}_0 \circ \bar{\xi}_1 \circ \ldots \circ \bar{\xi}_{i-1} R^i g = g^{2^i} \circ \bar{\xi}_0 \circ \ldots \circ \bar{\xi}_{i-1}$$

Define $D_{i,0} = \bar{\xi}_0 \circ \ldots \circ \bar{\xi}_{i-1}(B_0)$. It is invariant by g^{2^i} because

$$g^{2^i}(D_{i,0}) = g^{2^i} \bar{\xi}_0 \circ \ldots \circ \bar{\xi}_{i-1}(B_0) = \bar{\xi}_0 \circ \ldots \circ \bar{\xi}_{i-1} \circ R^i g(B_0) \subset D_{i,0}$$

Take

$$D_{i,k} = g^{l_0} \circ \bar{\xi}_0 \circ (Rg)^{l_1} \circ \bar{\xi}_1 \circ \ldots \circ (R^{i-2}g)^{l_{i-2}} \circ \bar{\xi}_{i-2} \circ (R^{i-1}g)^{l_{i-1}} \circ \bar{\xi}_{i-1}(B_0)$$

where $l_{i-1} \ldots l_2 l_1 l_0$ is the binary writing of the index k.$(0 \le k \le 2^i - 1)$

It is easy to check that $g^{2^i}(D_{i,k}) \subset D_{i,k}$. In fact $g(D_{i,k}) = D_{i,k+1}$ for $0 \le k \le 2^i - 2$, and $g(D_{i,2^i-1}) \subset D_{i,0}$.

Also, $\max_k \operatorname{diam}(D_{i,k}) \le \beta^i(\text{constant}) \to 0$ with $i \to \infty$.

From the definition of $D_{i,k}$, considering that $(R^i g)^{l_i} \bar{\xi}_i(B_0) \subset B_0$, we obtain that:

$$D_{i,k(mod 2^i)} \supset D_{i+1,k}$$

Let p_0^m be the fixed point of $R^m f$. Then

$$p_k^m = \xi_{R^m f} \circ \ldots \circ \xi_{R^{m+k-1} f} p_0^{m+k}$$

is a periodic point of period 2^k of $R^m f$.

Take $\bar{p}_k$ an accumulation point of $\{p_0^{m_j+k}\}_j$ with k fixed and $j \to \infty$. We have $\bar{p}_k \in D_{k,0}$ where $\bar{p}_k$ is a periodic point of period 2^k of g. The orbit by g of $\bar{p}_k$ has one point on each $D_{k,j}$ $j = 0, \ldots, 2^k - 1$, because $g(D_{k,j}) \subset D_{k,j+1(mod 2^k)}$.

$Dg^{2^k}(\bar{p}_k) = \lim_{j\to\infty} D(R^{m_j} f)^{2^k}(p_k^{m_j})$ has an eigenvalue ρ_k smaller or equal than -1, because of the continuity of the eigenvalues. As $\det Dg = 0$, the other eigenvalue is zero.

We have

$$Dg^{2^k}(\bar{p}_k)u_k = \prod_{i=0}^{2^k-1} Dg(g^i(\bar{p}_k))u_k$$

Thus $\dim(\ker Dg(g^i(\bar{p}_k))) = 1$ for all $k \ge 0$, and all $i = 0, 1, \ldots, 2^k - 1$ and also $\|Dg(g^{i(k)}(p_k))v_k\| \ge 1$ for some $i(k)$, and some unitary vector v_k.

Let us call $q_k = g^{i(k)}(\bar{p}_k)$. We have a sequence $\{(q_k, v_k)\}_{k\ge1}$, with $q_k \in B_0$, $\|v_k\| = 1$, $\|Dg(q_k)v_k\| > 1$. Let us take now a subsequence k_j such that (q_{k_j}, v_{k_j}) is convergent to (q, v). We have $\|Dg(p)u\| > \frac{1}{2}$ for (p, u) in a neighborhood of (q, v). This neighborhood is an open set $V \subset B_0$ and a cone of unitary vectors that are not contracted more than $\frac{1}{2}$ by $Dg(p)$.

Thus, for all $p \in V$, $\dim\ker(Dg(p)) = 1$. The unitary vector of $\ker Dg(p)$ define a vectorfield in V of class C^{r-2}. For $r \geq 3$, this vectorfield define a C^{r-2} foliation. The image by g of each leaf is a point, because the derivative of g along the leaf is zero.

Take a set D_{k_0,j_0} in V with k_0 sufficiently large so it is contained in a trivializing neighborhood of the foliation. Let us call D to the union of the leaves intersecting D_{k_0,j_0}. It is invariant by $g^{2^{k_0}}$. D has non void interior because D_{k_0,j_0} is connected and has points of period 2^k for all $k \geq k_0$, that can not be contained in the same leaf of the foliation. Take $k = k_0 + 1$. In the trivializing coordinates (x, y) in D, each leaf corresponds to constant x. As the image by g (and any of its iterates) of each leaf is a point, we have for (x, y) in D: $g^{2^k}(x, y) = (g_1(x), g_2(x))$. Let us see that g_1 has at least one critical point.

We have in D a fixed point $q_k = (x_k, y_k)$ of g^{2^k}, and a fixed point $q_{k-1} = (x_{k-1}, y_{k-1})$ of $g^{2^{k-1}}$. As $Dg^{2^k}(q_{k-1})$ has a eigenvalue greater or equal than one, it is obtained that $g_1'(x_k) \geq 1$. But, as $Dg^{2^k}(q_k)$ has a negative eigenvalue smaller or equal that -1, we have that $g_1'(x_{k-1}) \leq -1$. There must exist at least one point where $g_1' = 0$. $\square$

The last theorem asserts that dissipative C^r-cascades of period doubling in dimension two, verifying the hypothesis of the uniform bounds, are a perturbation of an one-dimensional multimodal map, when microscopically looked.

3 Approximation with homoclinic tangencies of the Gambaudo-Tresser n-dimensional cascade

The purpose of this section is to prove that the Gambaudo-Tresser [GT,1992] example of cascade of period doubling in dimension n is approximated with homoclinic tangencies. It is of type C^r with r increasing with n, and is not uniformly dissipative. Indeed at the points of the Cantor set the determinant of the jacobian matrix is one.

Theorem 3.1 (Gambaudo-Tresser, [GT,1992]) *For any $r > 1$ there exists $n \geq 2$ and a C^r-map of the n-dimensional ball that is a cascade of period doubling, whose Cantor set attractor contains an affine copy of itself scaled by a factor λ that can be chosen in an interval.*

Remark 3.2 As the geometry of the Cantor set can be chosen, this theorem implies that there is no hope of finding universal geometry of the Cantor set attractor. In other words this example can not be reducible to the Feigenbaum's one dimensional map.

The proof of the theorem is constructive. As we shall use later this construction, we include a summary of the proof of [GT,1992], mainly to fix the notation to be used later.

Proof: Let us define F_0, a C^∞ diffeomorphism in the unitary n-dimensional ball D verifying the following conditions:

a) F_0 is the identity in a thin shell $D \setminus D_{1-\gamma}$, where $D_{1-\gamma}$ is the ball of radius $1-\gamma$ concentric with D.

b) Consider 2^n disjoint balls $D_{1,i},\ i = 0,\ldots,2^n-1$ of radius $\lambda < 1$, contained in $D_{1-\gamma}$, leaving enough room to move rigidly any pair of these disjoint balls until they exchange their positions. It is enough that $\lambda < \frac{1-\gamma}{2\sqrt{n}+1}$

There is an isotopy $\{\psi_t\}_{t\in[0,1]}$ from the identity ψ_0 =id to $F_0 = \psi_1$; ψ_t restricted to $D_{1,i}$ for each $i = 0,\ldots 2^n-1$ is a traslation, and $F_0(D_{1,i}) = D_{1,i+1(mod 2^n)}$.

c) F_0 has one single periodic orbit of period $1,\ 2,\ldots,\ 2^{n-1}$ of saddle type of stable codimension one in $M_1 = D_{1-\gamma} \setminus \cup_{i=0}^{2^n-1} D_{1,i}$, and no other periodic orbits in M_1.

d) The set $\cup_{i=0}^{2^n-1} D_{1,i}$ is an attractor for F_0, while the shell $D \setminus D_{1-\gamma}$ is an attractor for the inverse mapping F_0^{-1}.

We then have that $F_0^{2^n}|_{D_{1,i}}$ is the identity.

Let us modify F_0 in $\cup_{i=0}^{2^n-1} D_{1,i}$ by self similarity to obtain F_1 such that the behavior of $F_1^{2^n}|_{D_{1,0}}$ copies that of F_0 in D. Let F_1 be defined as F_0 in $D \setminus \cup_i D_{1,i}$ and

$$F_1|_{D_{1,i}} = \Lambda_{1,i+1(mod 2^n)} \circ \psi_{\frac{i+1}{2^n}} \circ \psi^{-1}_{\frac{i}{2^n}} \circ \Lambda^{-1}_{1,i}$$

where $\Lambda_{1,i}$ is the homotecy transforming the ball D onto $D_{1,i}$ (the homotecy rate is λ). F_1 is of class C^∞ because on each ball $D_{1,i}$, F_0 and F_1 coincide in the shell $D_{1,i} \setminus \Lambda_{1,i}(D_{1-\gamma})$.

$F_1^{2^n}|_{D_{1,0}} = \Lambda_{1,0} F_0 \Lambda^{-1}_{1,0}$ because

$$F_1|_{D_{1,2^n-1}} \circ \ldots \circ F_1|_{D_{1,1}} \circ F_1|_{D_{1,0}} = \Lambda_{1,0} \circ \psi_1 \circ \Lambda^{-1}_{1,0} = \Lambda_{1,0} \circ F_0 \circ \Lambda^{-1}_{1,0}$$

For $i = 0,\ldots,2^n-1$, consider $\Lambda_{1,0}(D_{1,i})$. They are 2^n balls inside $D_{1,0}$ that are moved by translations with F_1 and its iterates, generating a family of 2^{2n} balls $D_{2,j},\ j = 0,\ldots,2^{2n}-1$ of radius λ^2 inside the balls $D_{1,i}$ for $i = 0,\ldots,2^n-1$. Now: $F_1(D_{2,j}) = D_{2,j+1(mod 2^{2n})}$ for $j = 0,\ldots,2^{2n}-1$ and $F_1^{2^{2n}}|_{D_{2,j}}$=id.

By induction, in the step $h \geq 1$ we modify F_{h-1} inside the 2^{hn} balls $D_{h,j},\ j = 0,\ldots,2^{hn}-1$ of radius λ^h. Having $F_{h-1}^{2^{hn}}|_{D_{h,j}}$=id, we construct F_h defined as follows:

$$F_h = F_{h-1} \text{ in } D \setminus \cup_j D_{h,j} \text{ and}$$

$$F_h\mid_{D_{h,j}}=\Lambda_{h,j+1(mod 2^{hn})}\circ\psi_{\frac{j+1}{2^{hn}}}\circ\psi^{-1}_{\frac{j}{2^{hn}}}\circ\Lambda^{-1}_{h,j} \tag{1}$$

where $\Lambda_{h,j}$ is the homotecy transforming the ball D onto $D_{h,j}$ (it is a homotecy of rate λ^h).

In this way we have defined in D a sequence of C^∞ maps $\{F_h\}_{h\geq 0}$. We claim that F_h is a Cauchy sequence in the topology C^r for certain r depending on n. So it defines a map F in the ball D, fixed by the renormalization $F=\Lambda^{-1}_{1,0}\circ F^{2^n}\circ\Lambda_{1,0}$.

In fact F_h-F_{h-1} is null in the complement of $\cup_{j=0}^{2^{hn}-1}D_{h,j}$ so they differ only in the 2^{hn} balls of radius λ^h that are interchanged both with F_h and F_{h-1}. Thus $\|F_h-F_{h-1}\|_{C^0}<2\lambda^h$ with $\lambda<1$.

Now, the derivatives of $F_{h-1}\mid_{\cup_j D_{h,j}}$ are the identity because it is a traslation restricted to each of the balls $D_{h,j}$. It is left to prove for h large enough that

$$\|(DF_h-\mathrm{id})\mid_{\cup_j D_{h,j}}\|_{C^{r-1}}<k\alpha^h$$

with some $\alpha<1$.

In fact, from (1)

$$\|DF_h-\mathrm{id}\|_{C^{r-1}}\leq(\lambda^{-h})^{r-1}\|D(\psi_{\frac{j+1}{2^{hn}}}\psi^{-1}_{\frac{j}{2^{hn}}})-\mathrm{id}\|_{C^{r-1}}$$

For the isotopy ψ_t we have a constant k such that

$$\|\psi_t\circ\psi_s^{-1}-\mathrm{id}\|_{C^r}<k|t-s|$$

for all t and s such that $t-s$ is small enough. So

$$\|D(\psi_{\frac{j+1}{2^{hn}}}\psi^{-1}_{\frac{j}{2^{hn}}})-\mathrm{id}\|_{C^{r-1}}\leq k\,\frac{1}{2^{hn}}$$

We thus have

$$\|DF_h-\mathrm{id}\|_{C^{r-1}}\leq k\left(\frac{1}{2^n\lambda^{r-1}}\right)^h$$

To have $\{F_h\}_{h\geq 1}$ a Cauchy sequence it is enough that $2^n\lambda^{r-1}>1$, that is, $n>-\frac{(r-1)\log\lambda}{\log 2}$. The interval in which λ can be chosen is $\frac{1-\gamma}{2\sqrt{n}+1}>\lambda>\frac{1}{2^{n/(r-1)}}$ for r,γ,n such that $2^{n/(r-1)}>\frac{2\sqrt{n}+1}{1-\gamma}$.

Now we have defined $F=\lim_{h\to\infty}F_h$ in the C^r topology. It is not still a cascade because it has a shell $D\setminus D_{1-\gamma}$ of fixed points and because of the self-similar construction it has shells inside the balls of generation h all formed by periodic points of period 2^h.

It is enough to change F_0 (and the isotopy ψ_t correspondingly) in a neighborhood of the shell $D\setminus D_{1-\gamma}$ so that D is mapped inside itself, and in a neighborhood of $\cup_{i=0}^{2^n-1}D_{1,i}$ so that the image of $D\setminus\cup_i D_{1,i}$ is not contained in itself. $\square$

Theorem 3.3 *Let F be the C^r cascade of period doubling in dimension n of the theorem of Gambaudo-Tresser above. Given $\epsilon > 0$ there exists G of type C^r, exhibiting a homoclinic tangency and such that $\|G - F\|_{C^r} < \epsilon$.*

Proof: Let $\{\psi_t\}_{0\le t\le 1}$ be the isotopy such that $\psi_0 =$id, $\psi_1 = F_0$ as in the proof of the last theorem. Define $\{\tilde{\psi}_t\}_{0\le t\le 1}$ such that $\tilde{\psi}_t = \psi_{2t}$ for $0 \le t \le \frac{1}{2}$, and $\{\tilde{\psi}_t\}_{\frac{1}{2}\le t\le 1}$ is the transformation $\delta_t \circ F_0$ where $\delta_{\frac{1}{2}}$ is the identity, and δ_t is constructed below.

Let V be a connected open set, disjoint with $\cup_i D_{1,i}$, that does not contain any periodic point of F_0 and such that $W^s(p_0) \cap V$ and $W^u(p_0) \cap V$ are contained in fundamental domains of $W^s(p_0)$ and $W^u(p_0)$ respectively.

Now δ_1 is a map that is the identity in the complement of V and takes the points of the arc $W^u(p_0) \cap V$ and pushes them to be tangent to $W^s(p_0)$. This can be done with an isotopy $\{\delta_t\}_{\frac{1}{2}\le t\le 1}$ with $\delta_{\frac{1}{2}} =$id, leaving fixed all the points of the complement of V. Now consider as in the proof of the last theorem the map F constructed as $\lim_{h\to\infty} F_h$ with F_h and F_{h-1} differing only in the balls $D_{h,j}$ of generation h. Let us define $\tilde{F}_h$ as follows:

$$\tilde{F}_h = \begin{cases} F & \text{in } D \setminus \cup_j D_{h,j} \\ \Lambda_{h,j+1(mod 2^{hn})} \circ \tilde{\psi}_{\frac{j+1}{2^{hn}}} \circ \tilde{\psi}^{-1}_{\frac{j}{2^{hn}}} \circ \Lambda^{-1}_{h,j} & \text{in } D_{h,j} \end{cases}$$

where $\Lambda_{h,j}$ is the homotecy transforming the ball D onto $D_{h,j}$. Now, by construction $\tilde{F}_h$ has a periodic point in $D_{h,0}$ of period 2^{hn} exhibiting a homoclinic tangency. It is left to show that $\|\tilde{F}_h - F\|_{C^r} \to_{h\to\infty} 0$.

$$\|\tilde{F}_h - F\|_{C^r} \le \|\tilde{F}_h - F_h\|_{C^r} + \|F_h - F\|_{C^r}$$

As $F = \lim_{h\to\infty} F_h$, the second term of the sum above is less that ϵ for h large enough. As $F_h = F = \tilde{F}_h$ in $D \setminus \cup_j D_{h,j}$, and they differ in the balls $D_{h,j}$ that are interchanged both with F_h and $\tilde{F}_h$, we have

$$\|\tilde{F}_h - F_h\|_{C^0} < 2\,\mathrm{diam} D_{h,j} = 2\lambda^h \to_{h\to 0} 0$$

Now:

$$\|D(\tilde{F}_h - F_h)\|_{C^r-1} = \|D\tilde{F}_h - \mathrm{id}\|_{C^{r-1}} + \|DF_h - \mathrm{id}\|_{C^{r-1}} \le$$

$$\le \frac{1}{(\lambda^h)^{r-1}} \|D(\tilde{\psi}_{\frac{j+1}{2^{hn}}} \circ \tilde{\psi}^{-1}_{\frac{j}{2^{hn}}}) - \mathrm{id}\|_{C^{r-1}} + k\left(\frac{1}{2^n \lambda^{r-1}}\right)^h$$

For the isotopy $\tilde{\psi}$ we have a constant $\tilde{k}$ such that

$$\|\tilde{\psi}_t \circ \tilde{\psi}_s^{-1} - \mathrm{id}\|_{C^r} < \tilde{k}|t-s| \text{ for all } t \text{ and } s$$

So

$$\|D(\tilde{F}_h - F_h)\|_{C^{r-1}} \leq \frac{k+\tilde{k}}{2^{hn}} \frac{1}{(\lambda^h)^{r-1}} = (k+\tilde{k}) \left(\frac{1}{2^n \lambda^{r-1}} \right)^h \to_{h\to\infty} 0$$

Thus $\|\tilde{F}_h - F\|_{C^r} \to_{h\to\infty} 0$ as wanted. □

References

[BGLT,1993] C.Bonatti, J.M.Gambaudo, J.M.Lion, C.Tresser: *Wandering domains for infinitely renormalizable diffeomorphisms of the disk.* Preprint Inst. Non Lineaire de Nice 1993

[C,1995] E.Catsigeras: *Cascades of period doubling of stable codimension one.* Tese do IMPA. (Rio de Janeiro) 1995

[CEK,1981] P.Collet, J.P.Eckmann, H.Koch: *Period doubling bifurcations for families of maps on R^n.* Journ. of Stat. Phys., 25, N.1 (1–14) 1981

[CT,1978] P.Coullet, C.Tresser: *Itérations d'endomorphisms et groupe de rénormalisation.* C.R. Acad.Sci.Paris 287 (577–588) 1978

[F,1978] M.J.Feigenbaum: *Quantitative universality for a class of nonlinear transformations.* Journ. Stat. Physics, vol 19 (25–52) 1978

[GT,1992] J.M.Gambaudo,C.Tresser: *Self similar constructions in smooth dynamics: rigidity, smothness and dimension.* Comm. in Math. Physics 150 (45–58) 1992

[M,1991] J.C.Martín-Rivas: *Homoclinic bifurcations and cascades of period doubling in higher dimensions.* Tese do IMPA. (Rio de Janeiro) 1991

[MV,1993] L.Mora, M.Viana: *Abundance of Strange Attractors.* Acta Math., vol 171 (1–71) 1993

[N,1979] S.Newhouse: *The abundance of wild hyperbolic sets and nonsmooth stable sets for diffeomorphisms.* Publ. Math. I.H.E.S. 50 (101–151) 1979

[P,1991] J.Palis: *Homoclinic Bifurcations, Sensitive Chaotic Dynamics and Strange Attractors.* Dynamical Systems and Related Topics, World Scientific (466–472) 1991

[PT,1993] J.Palis, F. Takens: *Hyperbolicity and sensitive chaotic dynamics at homoclinic bifurcations.* Cambridge University Press 1993

[S,1991] D.Sullivan: *Bounds, quadratic differentials and renormalization conjectures.* Mathematics into the twenty first century Vol.2 AMS 1991

[YA,1983] J.A.Yorke, K.T.Alligood: *Cascades of period doubling bifurcations: a prerrequisite for horseshoes.* Bull.A.M.S. 9 (319–322) 1983