Conselho Nacional de Desenvolvimento Científico e Tecnológico
Instituto de Matemática Pura e Aplicada

INFORMES DE MATEMÁTICA

Série F-077/95

CASCADES OF PERIOD DOUBLING OF STABLE CODIMENSION ONE

Eleonora Castsigeras

Rio de Janeiro
Março/95

Conselho Nacional de Desenvolvimento Científico e Tecnológico
Instituto de Matemática Pura e Aplicada

CASCADES OF PERIOD DOUBLING OF STABLE CODIMENSION ONE

Eleonora Catsigeras

Tese apresentada para obtenção do Título de Doutor em Ciências

Rio de Janeiro
Fevereiro/1995

CASCATAS DE DUPLICAÇÃO DE PERÍODO DE CODIMENSÃO UM

Eleonora Catsigeras

Resumo

Considera-se as cascatas de duplicação de período em dimensão n cujos pontos periódicos têm codimensão estável um. Prova-se resultados de redução de dimensão em duas etapas: primeiro à dimensão dois e depois à dimensão um para as cascatas uniformemente dissipativas com geometria limitada. Obtém-se teoremas de aproximação por tangências homoclínicas, respectivamente para um exemplo de Gambaudo e Tresser, e para as cascatas que são perturbações analíticas do mapa de Feigenbaum em dimensão n.

Agradeço

ao meu orientador Jacob Palis pelos seus ensinamentos, o seu entusiasmo e o seu apoio,

ao professor Jorge Lewowicz,

aos professores Konstantin Khanin, Wellington de Melo e Marcelo Viana pelas sugestões e comentários,

a Carlos Morales, Enrique Pujals, Eduardo Colli e Heber Enrich que me ajudaram na elaboração da tese,

ao IMPA pela sua hospitalidade e pelo excelente nível científico,

ao CNPq do Brasil, à Faculdade de Engenharia do Uruguai e à UNESCO, pelo apoio financeiro,

e aos meus colegas do IMPA.

Introdução

O desdobramento de uma tangência homoclínica em uma família a um parâmetro de mapas de classe C^k ($k \geq 3$) localmente dissipativos em dimensão dois origina fenômenos dinâmicos importantes [NPT,1983] [PT,1987] [T,1986] [PT,1993]. Por exemplo: ferraduras e conjuntos hiperbólicos, cascatas de duplicação de período [YA,1983], mapas com infinitos poços [N,1979], atratores do tipo de Hénon [MV,1992] [BC,1991]. Alguns destes resultados tambem são válidos em dimensões maiores [PV,1991] [V,1991] [R,1992] [M,1991]. Em outras palavras, as famílias que desdobram uma tangência homoclínica apresentam muitas das bifurcações globais conhecidas. São exemplos notáveis de sistemas dinâmicos globalmente inestáveis.

Não é conhecido se as bifurcações homoclínicas são necessárias para a inestabilidade global. Precisamente, J.Palis [PT,1993] tem formulado a seguinte:

Conjetura

O subconjunto $\mathcal{H}$ de difeomorfismos que são hiperbólicos (i.e. com conjunto limite hiperbólico e sem ciclos) e dos que apresentam uma bifurcação homoclínica é denso no espaço de difeomorfismos de superficie de classe C^∞.

Ao formular a conjetura, J. Palis também apresenta o seguinte programa: tratar de aproximar com bifurcações homoclínicas os casos particulares de inestabilidade global, como por exemplo:

1. os difeomorfismos que têm um atrator na acumulação de bifurcações de duplicação de período (descoberto por Feigenbaum e independentemente por Coullet e Tresser [F,1978] [CT,1978]),

2. os difeomorfismos com um atrator de tipo de Hénon,

3. os difeomorfismos que exibem infinitos poços simultâneos.

Consideramos o primeiro caso do programa acima. Primeiro, na seção 1 definimos os mapas em R^n que exibem uma cascata de duplicação de período, os quais serão estudados ao longo do trabalho. Na seção 2 incluímos alguns resultados que tratam da redução de dimensão. Fazemos esta redução em duas etapas: primeiro à dimensão dois e depois à dimensão um. Sob hipótese de dissipatividade uniforme e existência de limites na geometria e na dinâmica do sistema, obtemos que as cascatas bidimensionais são perturbações de mapas em dimensão um que têm pelo menos um ponto crítico.

Na seção 3 estudamos um exemplo de Gambaudo e Tresser [GT,1992] de cascata de duplicação de período em n dimensões que contém uma cópia afim de si mesma e não é redutível à dimensão um. Aproximamos a cascata desse exemplo com mapas que exibem uma tangência homoclínica.

Finalmente, na seção 4 estudamos as cascatas de duplicação de período analíticas reais numa vizinhança do mapa de Feigenbaum en n dimensões. Definimos primeiro uma renormalização aplicável às perturbações do mapa de Feigenbaum. Trabalhamos com a topologia analítica para poder usar a diferenciabilidade da renormalização e obter as propriedades espectrais de sua derivada. Estas propriedades são estudadas aplicando-se um teorema de Collet, Eckmann e Koch [CEK,1981]. Modificamos a renormalização usada nesse teorema, alterando a mudança de variáveis, para encaixar bem com a teoría em dimensão um. Pode-se assim aplicar um resultado de Eckmann e Wittwer [EW,1987] que trata dos mapas unidimensionais. Nele obtém-se, perto do mapa de Feigenbaum, mapas unimodais cujo ponto crítico, ao ser iterado um número finito de vezes, vai para um repulsor periódico. Concluímos que os mapas analíticos n dimensionais, proximos ao de Feigenbaum, que são infinitamente renormalizáveis, são acumulados por mapas que exibem tangências homoclínicas.

CASCADES OF PERIOD DOUBLING OF STABLE CODIMENSION ONE

Eleonora Catsigeras

Introduction

The one-parameter unfolding of a homoclinic tangency, for locally dissipative maps of class C^k ($k \geq 3$), in dimension two, originates important dynamical phenomena [NPT,1983] [PT,1987] [T,1986] [PT,1993]. For instance: horseshoes and hyperbolic sets, cascades of period doubling bifurcations [YA,1983], maps with infinitely many sinks [N,1979], Hénon-like attractors [MV,1992] [BC,1991]. Some of the results are also valid in higher dimensions [PV,1991] [V,1991] [R,1992] [M,1991].

In other words, the families unfolding a homoclinic tangency have many of the known global bifurcations. They are notable examples of global instable systems. It is not known if the homoclinic bifurcations are in general necessary for global instability. Precisely, J. Palis had formulated the following:

Conjecture: *The subset $\mathcal{H}$ of diffeomorphisms that are either hyperbolic (i.e. with hyperbolic limit set and no cycles) or homoclinic bifurcating is dense in the space of C^∞ surface diffeomorphisms.* [PT,1993].

When formulating the question, J. Palis has also presented the following program: try to approximate with homoclinic bifurcations some particular global instabilities, as for example:

1. diffeomorphisms having an attractor (as discovered by Feigenbaum and independently by Coullet and Tresser [F,1978] [CT,1978]), at the accumulation of period doubling bifurcations.

2. diffeomorphisms having a Hénon-like attractor .

3. diffeomorphisms exhibiting infinitely many coexisting sinks.

We addresss at the first case of the program above.

First, in section 1, we define the maps in R^n exhibiting a cascade of period doubling that are studied along the work. In section 2 we include some results dealing with the reduction of the dimension. Under assumptions of uniform dissipativeness and boundness of the dynamical geometry, we obtain that the cascades are perturbations of one dimensional maps having at least one critical point.

In section 3 we study a particular example of Gambaudo and Tresser [GT,1992] of cascade of period doubling in n dimensions that contains a rescaled copy of itself, and is not reductible to dimension one. We approximate the cascade in this example with homoclinic bifurcating maps.

Finally, in section 4 we study the analytic cascades of period doubling appearing when perturbing the Feigenbaum's map in n dimensions. We define a renormalization that is applicable to the perturbations, working with the analytic topology to be able to explote the differentiability of this renormalization, and to obtain good spectral properties. These are studied applying a theorem of Collet, Eckmann and Koch [CEK,1981]. We modify the renormalization used in that work, moving the change of variables to fit well with the one-dimensional theory. We can thus apply a result of Eckmann and Wittwer [EW,1987] dealing with one-dimensional maps near the Feigenbaum's whose critical point lands, after finitely many iterates in a repellor. We conclude that the analytic n-dimensional maps near the Feigenbaum's that are infinitely renormalizable are accumulated with maps exhibiting an homoclinic tangency.

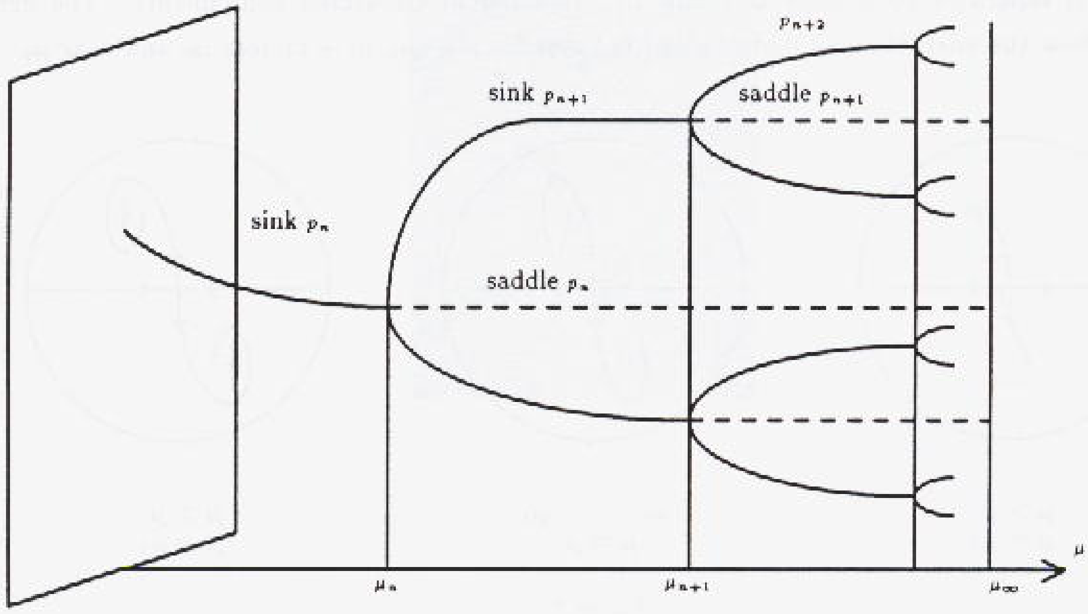


Figure 1

1 Definitions and general assumptions

Let B_0 be a domain in R^n (i.e. a connected and bounded set that is the closure of its interior) and I be a closed bounded interval in R.

Let $\{ f_\mu \}_{\mu\in I}$, $f_\mu : B_0 \mapsto \text{int}(B_0)$ be a one-parameter family of C^r maps ($r \geq 2$) such that for a monotone sequence $\mu_n \to \mu_\infty$ in I, f_μ exhibits a biffurcation of period doubling (from period 2^n to period 2^{n+1} at $\mu = \mu_n$, for integer $n \geq 0$). Let us denote as $p_n(\mu)$ a point of the orbit of period 2^n, that is a hyperbolic sink for $\mu_{n-1} < \mu < \mu_n$ and a hyperbolic saddle of stable codimension one for all $\mu > \mu_n$ in I. The derivative of $f^{2^n}_{\mu_n}$ at p_n has a single eigenvalue -1 and the other are strictly smaller than 1 in modulus.

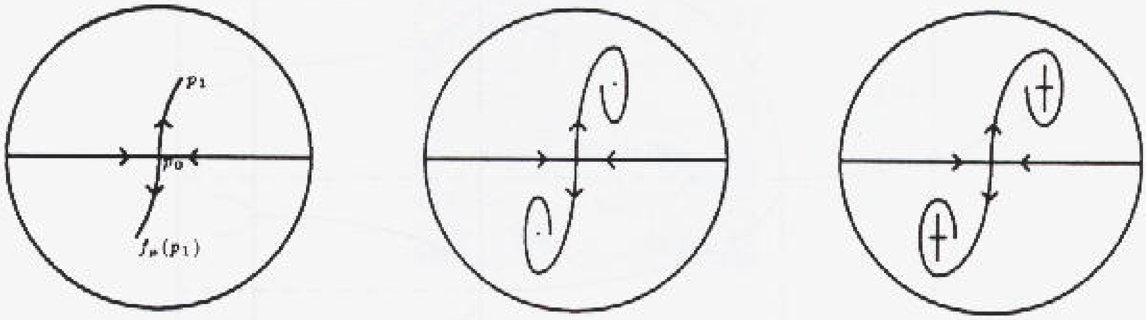
For $\mu \in (\mu_n, \mu_{n+1})$ there is a hyperbolic sink p_{n+1} of period 2^{n+1}, that becomes a saddle in $\mu = \mu_{n+1}$ and generates a period 2^{n+2} orbit. See the figure 1.

There are no other periodic points and no other biffurcations for $\mu > \mu_0$, $\mu \leq \mu_\infty$ except those described above.

For $\mu > \mu_0$, $\mu \leq \mu_\infty$ we assume that there exist two disjoint domains $B_{1,0}$ and $B_{1,1}$ contained in $\text{int}(B_0) \setminus W^s(p_0)$, depending continously on μ, that are forward invariant by f^2_μ, and $f_\mu(B_{1,j}) \subset$

$B_{1,j+1(mod 2)}$.

The ω-limit of all the orbits, except those in $W^s(p_0)$, is contained in $B_{1,0} \bigcup B_{1,1}$. Thus $W^s(p_0(\mu))$ separates B_0 leaving $B_{1,0}$ and $B_{1,1}$ in different connected components. The figure 2 shows how the unstable manifold of $p_0(\mu)$ behaves for $\mu \in (\mu_0, \mu_1 + \epsilon)$ near μ_0 and near μ_1.



$\mu > \mu_0$
$\mu \approx \mu_0$

$\mu_0 < \mu < \mu_1$
$\mu \approx \mu_1$

$\mu > \mu_1$
$\mu \approx \mu_1$

Figure 2

In other words after the bifurcation produced at μ_0 the behavior in $B_0 \setminus B_{1,0} \cup B_{1,1}$ survives for all $\mu > \mu_0$, $\mu \leq \mu_\infty$.

The assumptions above are supposed to hold also for $n \geq 1$, $\mu > \mu_n$, $p_n(\mu)$ instead of p_0, and domains $B_{n,j} \subset \text{int} B_{n-1,j(mod 2^{n-1})}$ instead of $B_{1,j} \subset \text{int} B_0$, and $f_\mu^{2^n}$ instead of f_μ. The sets $B_{n,j}$ are disjoint for fixed n and $0 \leq j \leq 2^n - 1$ and will be called *domains of generation n*.

For simplicity we leave out in the notation the parameter μ, when working at μ_∞.

In the tree diagram of bifurcations for the one-parameter family we can choose, at each bifurcation, one of the two points of the double period orbit. A *route* is a sequence $\{i(n)\}_{n \geq 1}$ with $i(n) \in \{0, 1\}$ corresponding to the choice at each bifurcation of one of the two branches in the tree diagram.

To each route corresponds a sequence of contained domains:

$$B_0 \supset B_{1,j(1)} \supset \ldots \supset B_{n,j(n)} \supset \ldots$$

where $j(n) = \sum_{k=1}^{k=n} i(k) 2^{k-1}$. Observe that $0 \leq j(n) \leq \sum_{k=1}^{k=n} 2^{k-1} = 2^n - 1$

Let us denote with J the set of all sequences $\{j(n)\}_{n \geq 1}$ constructed as above, each sequence for each route. For each route corresponds a compact subset $\cap_{n=1}^{\infty} B_{n,j(n)}$.

We have the compact set:

$$K = \bigcup_{\{j(n)\}\in J} \bigcap_n B_{n,j(n)} = \bigcap_n \bigcup_j B_{n,j}$$

The last equality is due to the fact that the domains of generation n are all disjoint. We have that $f(K) \subset K$.

We assume that all the routes are convergent to a point. In other words any choice of the branches in the tree diagram converge to a single point for $\mu \to \mu_\infty$, or $\# \cap_n B_{n,j(n)} = 1$ for any sequence $\{j(n)\} \in J$.

In [BGLT,1993] are studied the wandering domains obtained without the convergence assumption. In particular they find a C^1 example in which K contains a wandering domain (i.e with non void interior). Also they prove that for $C^{1+\alpha}$ diffeomorphisms with a hypothesis of bounded geometry, the connected components of K (that are not necessarily points) have zero Lebesgue measure.

Proposition 1.1 *Let $f_\mu \to f$ be a family verifying all the assumptions above. Then K is an invariant Cantor set and there is uniform convergence for all the routes, i.e. given ϵ there exists N such that $diam(B_{n,j}) \leq \epsilon$ for all $n > N$, and all $0 \leq j \leq 2^n - 1$.*

Proof: By contradiction let us suppose that there exists $\epsilon > 0$ such that for any N there exist $n > N$ and $j \in \{0, 1, \ldots, 2^n - 1\}$ for which $\mathrm{diam} B_{n,j} > \epsilon$. As any domain of generation n is contained in a domain of generation $n - 1$, we have a sequence of domains $B_{n,j(n)}$ of diameter greater then ϵ. This sequence can not be formed of domains each containig the following, because by hypothesis any route is convergent to a point. In other words $\{j(n)\} \notin J$.

Let us consider $B_{n,j(n)}; B_{n+k,j(n+k)}$ for some fixed $n \geq 1$, and fixed $k \geq 1$. As $B_{n+k,j(n+k)}$ is contained in some ball of generation n, say $B_{n,j(n,k)}$ we can substitute $B_{n,j(n)}$ to have

$$B_{n,j(n,k)} \supset B_{n+k,j(n+k)}$$

For fixed n, the infinite sequence $\{j(n,k)\}_{k\geq 1}$ takes its values in a finite set: $0 \leq j \leq 2^n - 1$. Thus there exists some value j_0 that is repeated infinite times, for values of k in $\{k_i\}_{i\geq 1}$ and n fixed. Fix j_o; we obtain:

$$B_{n,j_0} \supset B_{n+k_1,j(n+k_1)}, B_{n+k_2,j(n+k_2)}, \ldots, B_{n+k_i,j(n+k_i)}, \ldots$$

Now we change $j(n + k_1)$ properly as above, such that, for a new sequence $\{k_{i_l}\}_{l\geq 1}$:

$$B_{n,j_0} \supset B_{n+k_1,j_1} \supset B_{n+k_{i_1},j(n+k_{i_1})}, \ldots B_{n+k_{i_2},j(n+k_{i_2})}, \ldots$$

In this way it is constructed a sequence of domains, each containing the following, all with diameter greater than ϵ, contradicting the hypothesis of convergence of the routes. $\square$

Definition 1.2 A *cascade of period doubling* is a map $f : B_0 \subset R^n \mapsto \text{int } B_0$ provided with a family of domains $B_{m,j}$, $m \geq 0$, $0 \leq j \leq 2^m - 1$, $B_{0,0} = B_0$ such that

a) $B_{m,j} \cap B_{m,k} = \emptyset$ for $j \neq k$ and $B_{m,j} \subset \text{int } B_{m-1,j \,(mod\, 2^{m-1})}$ for $m \geq 1$.

b) diam $B_{m,j} \to 0$ with $m \to \infty$ uniformly in j.

c) $f(B_{m,j}) \subset B_{m,j+1\,(mod\, 2^m)}$ and $f^{2^m}(B_{m,j}) \subset \text{int } B_{m,j}$.

d) There exists a periodic hyperbolic orbit of saddle type, of stable codimension one with negative expansing eigenvalue, of period 2^m with one point in $B_{m,j}$ for each $j = 0, 1, \ldots, 2^m - 1$, and there are no other periodic points.

e) For any $q \in B_0$ and all $m \geq 0$ the ω-limit of q is contained in the union of the periodic orbits of period $1, 2, \ldots, 2^{m-1}$ with $\cup_{j=0}^{2^m-1} \text{int} B_{m,j}$.

For instance if $\{f_\mu\}_{\mu \in I}$ is a family verifying all the assumptions at the beginning of this section, then $f_{\mu_\infty} = \lim_{\mu \to \mu_\infty} f_\mu$ is a cascade of period doubling.

The definition above implies that the periodic orbit of period 2^m and its stable manifold are disjoint with $\cup_{j=0}^{2^{m+1}-1} B_{m+1,j}$.

Let p_m be the periodic point of period 2^m in $B_{m,0}$. The points of $B_{m,0} \setminus W^s(p_m)$ are classified in two sets: those points having an iterate by $f^{2^{m+1}}$ (and all of its following iterates) in int$(B_{m+1,0})$ and those having an iterate in int$B_{m+1,2^m}$. They are open sets. Thus $W^s(p_m)$ disconnects the domain $B_{m,0}$ leaving $B_{m+1,0}$ and $B_{m+1,2^m}$ in different connected components.

Remark 1.3 We will also have a cascade of period doubling given f provided not with the whole family of the domains $B_{m,j}$ of all the generations, but only with a subfamily $\{B_{m_k,j}, 0 \leq j \leq 2^{m_k} - 1\}, m_k \to \infty$ of the domains of generations m_k.

Each set $B_{m_k,j}$ must be provided with a saddle type point p_{m_k} of period 2^{m_k} and 2^i points of period 2^{m_k+i} for $0 < i < m_{k+1} - m_k$.

The properties a) to e) must be fulfilled for $B_{m_k,j}$ and all the periodic points mentioned above. Thus domains of generation $m_k + i$ can be constructed as follows: start with $i = m_{k+1} - m_k - 1$. Suppose constructed the domains of generation $m_k + i + 1$. Take a periodic point p_{m_k+i} of period 2^{m_k+i} and choose some disjoint neighborhoods V_i of p_{m_k+i} and U_j of the domains $B_{m_k+i+1,j}$. When iterating V_i by $f^{2^{m_k+i}}$ it approaches the unstable manifold, which is attracted by $\cup_j B_{m_k,j}$, so its two branches enter U_{j_1} and U_{j_2}. Take the finite union of U_{j_1}, U_{j_2} and the finite number of iterates of V_i until they intersect U_{j_1} and U_{j_2}. Its closure will be the domain of generation m_k+i containing p_{m_k+i}.

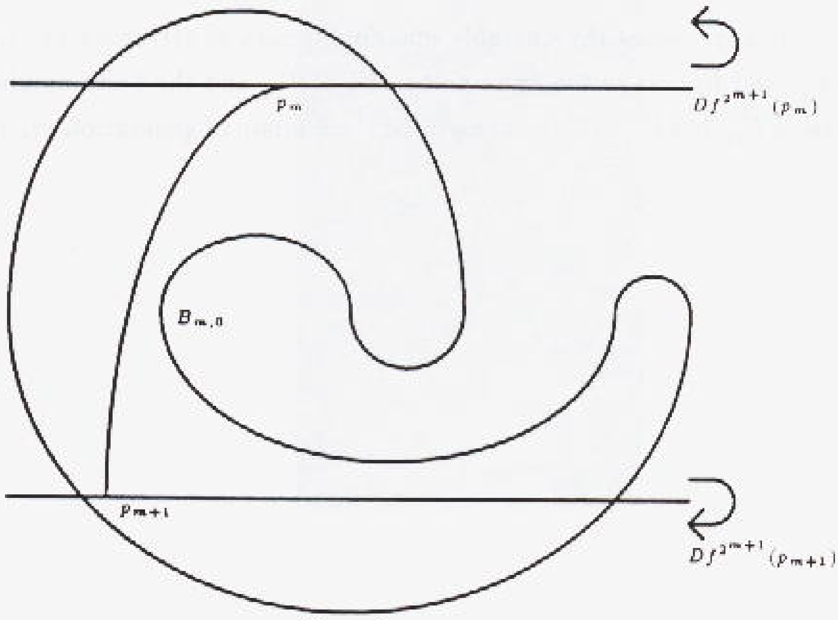


Figure 3

2 Reduction of the dimension

Our purpose in this section is to reduce the dimension. We do that in two steps:

First, when there is a contractive invariant foliation of codimension 2, we can reduce f to a 2-dimensional cascade. This is done in proposition 2.2. Unfortunately there is not an invariant foliation of codimension one, when working with diffeomorphisms that exhibit a cascade of period doubling, as shown below in proposition 2.1.

Second, we reduce 2-dimensional cascades to 1-dimensional ones, under some hypothesis of area contractiveness, and of uniform bounds of the sucessive renormalizations. This is done in theorem 2.4.

Proposition 2.1 *Let $f : B_0 \subset R^n \to f(B_0) \subset B_0$ be a C^r diffeomorphism ($r \geq 1$) that is a cascade of period doubling. Then it does not exist a C^1 contractive foliation invariant by f of codimension one.*

Proof: By contradiction suppose that there exists an invariant C^1 foliation. Consider f^{2^m} with m sufficiently large to have $B_{m,0}$ contained in a neighborhood where the foliation can be trivialized. After the trivialization each of the leaves of the foliation is a subspace of codimension one, which we draw horizontal in the figure 3, and that separates R^n in lower and upper semispaces. In $B_{m,0}$ there is a fixed point p_m of f^{2^m} and a period 2 point p_{m+1}, both with negative expansive eigenvalues of $Df^{2^m}(p_m)$ and $Df^{2^{m+1}}(p_{m+1})$ respectively. Connect p_m and p_{m+1} with a continuous curve inside $B_{m,0}$. Applying $Df^{2^{m+1}}$ to any leaf of the foliation intersecting $B_{m,0}$ we obtain another leaf intersecting $B_{m,0}$. Call H to the set of points of the curve such that $Df^{2^{m+1}}$ maps the upper semispace onto the upper semispace. For instance p_m is in H. Call K to the set of points where the derivative maps the upper subspace onto the lower subspace. The point p_{m+1} is in K. As the map f is a diffeomorphism the sets H and K are open and complementary in the curve. This contradicts the connectedness of the curve. □

Proposition 2.2 (Reduction to dimension 2) *Let $f : B_0 \subset R^n \mapsto f(B_0) \subset B_0$ be a C^r diffeomorphism ($r \geq 1$) that is a cascade of period doubling.*

Suppose that there exists in B_0 a C^r foliation that is f-invariant and contractive by f, of codimension two.

Then there exist an integer m, a domain B_m invariant by f^{2^m} and a system of coordinates $\{(x,y) : x \in R^2, y \in R^{n-2}\}$ in B_m such that in those coordinates

$$f^{2^m}(x,y) = (g(x), h(x,y))$$

where g is a cascade of period doubling in dimension 2.

Proof: Let us call $B_{m,j}, 0 \leq j \leq 2^m - 1$ the domains of generation m in the definition of cascade. They are invariant by f^{2^m}. Take m large enough such that $B_{m,0}$ is contained in the domain of a trivializing chart of the given foliation. In such trivializing coordinates $\{(x,y) : x \in R^2, y \in R^{n-2}\}$ the leaves are obtained fixing x. As they are invariant by f, we have:

$$f^{2^m}(x,y) = (g(x), h(x,y))$$

We call B_m to $B_{m,0}$.

We shall see that g is a cascade. Let us call Π to the projection on the first coordinate plane. (It is the projection along the leaves of the foliation). Define $\tilde{B}_0 = \Pi(B_m)$. It contains the point $x_0 = \Pi(p_m)$, where p_m is the periodic point of f in B_m with period 2^m. As $f^{2^m}(x_0, y_0) = (g(x_0), h(x_0, y_0)) = (x_0, y_0)$, we have that x_0 is a fixed point of g. It is hyperbolic of saddle type with stable codimension one and negative expansive eigenvalue: in fact the matrix $Df^{2^m}(x_0, y_0)$ is triangular and thus the eigenvalues of Df^{2^m} are those of Dg and those of $D_y h$. The last ones are in modulus smaller than one because the foliation is contractive. Then the expansive eigenvalue of Df^{2^m} is found in Dg.

The stable manifold of p_m contains all the leaves of the foliation that intersects, because the foliation is contractive with f, and thus for any q in the leaf through $p \in W^s(p_m)$:

$$\|f^{j2^m}(q) - f^{j2^m}(p)\| \to 0 \text{ with} j \to \infty$$

As $f^{j2^m}(p) \to p_m$ with $j \to \infty$ it is obtained that $f^{j2^m}(q) \to p_m$. So $q \in W^s(p_m)$.

Then $\Pi(W_s(p_m))$ is a curve in $\tilde{B}_0$. It is the stable manifold of x_0, because if $f^{2^m j}(q) \to p_m$ then

$$g^j(\Pi(q)) = \Pi f^{2^m j}(q) \to \Pi(p_m) = x_0$$

The subdomains $\tilde{B}_{1,0}$ and $\tilde{B}_{1,1}$ for g are defined as the projections of the two subdomains of f of generation $m+1$ contained in B_m. They are disjoint: in fact, by contradiction suppose that one leaf of the foliation intersects the two subdomains of f of generation $m+1$ in points q_1 and q_2 respectively. As they are in the same contractive leaf, when iterating them by $f^{2^{m+1}}$ their distance converges to zero. But they are in different subdomains of generation $m+1$, that are invariant by $f^{2^{m+1}}$, closed and disjoint, thus having a positive distance.

All the points of $\tilde{B}_0 \setminus W^s(x_0)$ when sufficiently iterated by g land in $\tilde{B}_{1,0} \cup \tilde{B}_{1,1}$ because they are the projections of points of $B_m \setminus W^s(p_m)$. Thus there are not other fixed points of g besides x_0.

Analogously are constructed the periodic orbits of higher period for g and the domains of higher generation, projecting those of f. Finally all the routes of g are convergent because they are the projection of convergent routes of f. $\Box$

Definition 2.3 A C^r cascade of period doubling $f : B_0 \subset R^n \mapsto B_0$ is (doubling) *renormalizable* if the domain $B_{1,0} \subset \text{int} B_0$ invariant by f^2, is C^r diffeomorphic to B_0.

Calling $\xi_f : B_0 \mapsto B_{1,0}$ the diffeomorphism between the domains B_0 and $B_{1,0}$, we define *the renormalized* of f as

$$Rf = \xi_f^{-1} \circ f \circ f \circ \xi_f : B_0 \mapsto B_0$$

Observe that Rf is also a cascade of period doubling. By induction we define: a C^r map is *m times renormalizable* if it is $m-1$ times renormalizable and its $m-1$ renormalized is also renormalizable. In that case the m renormalized is:

$$R^m f = \xi_{R^{m-1}f}^{-1} \circ R^{m-1} f \circ R^{m-1} f \circ \xi_{R^{m-1}}$$
$$= \xi_{R^{m-1}f}^{-1} \circ \ldots \circ \xi_f^{-1} \circ f^{2^m} \circ \xi_f \circ \ldots \circ \xi_{R^{m-1}f}$$

A map is *infinitely renormalizable* if it is m times renormalizable for all m.

The definitions above are general. In particular cases we will impose some bounding conditions to the change of variables ξ_f. Normally they are affine transformations.

Theorem 2.4 (Reduction of dimension 2 to dimension 1) *Let $f : B_0 \mapsto B_0 \subset R^2$ be a C^r $(r \geq 3)$ cascade of period doubling in two dimensions that is C^r infinitely renormalizable and such that $R^m f$ is C^r bounded for all $m \geq 1$.*

Suppose that:

f is uniformly dissipative (i.e.: $0 \leq detDf(p) \leq \alpha < 1$ for all $p \in B_0$).

The changes of variables $\xi_{R^{m-1}f}$ used to define the m renormalized of f, are C^r bounded uniformly in m, and there exist $\beta < 1$ and $\gamma > 0$ independent of m such that

$$\max\{||D\xi_{R^m f}||_{C^0}, ||D(R^m f \circ \xi_{R^m f})||_{C^0}\} \leq \beta < 1$$

and

$$|det(D\xi_{R^m f})| \geq \gamma$$

Then:

There exists a C^{r-1} map g defined in B_0 such that:

For any given $\varepsilon > 0$ there exists an integer m verifying $||g - R^m f||_{C^{r-1}} \leq \varepsilon$,

There exists an iterate 2^k of g, a subset $D \subset B_0$ invariant by g^{2^k} and a C^{r-2} change of coordinates in D such that

$$g^{2^k}(x, y) = (g_1(x), g_2(x))$$

where g_1 is a multimodal (i.e. with at least one critical point) map in the interval.

Proof: Applying the Arzela-Ascoli theorem to the family of maps $R^m f$, there exists a subsequence m_j such that the $\lim_{j\to\infty} R^{m_j} f$ exists in the C^{r-1} topology. Let us call g to that limit.

We have:

$$\det Dg(q) = \lim_{j\to\infty} \det DR^{m_j} f(q)$$

$$\det DR^m f(q) = \prod_{i=0}^{m-1} \det D\xi_{R^i f}(q_{i+1}) \det Df^{2^m}(q_0) \prod_{i=0}^{m-1} \det(D\xi_{R^i f}(\tilde{q}_{i+1}))^{-1}$$

where $q_m = q$, $q_i = \xi_{R^i f} \circ \ldots \circ \xi_{R^{m-1} f}(q)$, $i = 0, \ldots, m-1$ and $\tilde{q}_0 = f^{2^m}(q_0)$, $\tilde{q}_{i+1} = \xi^{-1}_{R^i f} \circ \ldots \circ \xi_f(\tilde{q}_0)$, $i = 0, \ldots, m-1$. Thus

$$|\det D(R^m f)(q)| \leq \alpha^{2^m} a^m \to 0 \text{ with } m \to \infty$$

where $a > 1$ is a uniform bound of the jacobians of $\xi_{R^i f}$ and $\xi^{-1}_{R^i f}$, and $\alpha < 1$ is the bound of the hypothesis of uniform dissipativeness. Thus $\det Dg(q) = \lim_{j\to\infty} \det(DR^{m_j} f)(q) = 0$.

We already have a map g such that $\det Dg = 0$ in B_0, and $||g - R^{m_j}f||_{C^{r-1}} < \varepsilon$, for any given $\varepsilon > 0$, for all j sufficiently large, depending on ε.

As $\xi_{R^m f}$ and $R^m f$ are bounded uniformly in m in the C^r topology, there exist sucessive subsequences $\{m_j\}$ obtained making at each step i, $\xi_{R^{m_j+i}f}$ and R^{m_j+i} convergent in the C^{r-1} topology with i fixed and $j \to \infty$. Take the diagonal subsequence and then define, for each i:

$$R^i g = \lim_{j\to\infty} (R^{i+m_j}(f))$$

$$\bar{\xi}_i = \lim_{j\to\infty} \xi_{R^{m_j+i}f}$$

From

$$\xi_{R^{m_j}f} \circ \xi_{R^{m_j+1}f} \circ \ldots \xi_{R^{m_j+i-1}f} \circ R^{m_j+i}f = (R^{m_j}f)^{2^i} \circ \xi_{R^{m_j}f} \circ \xi_{R^{m_j+1}f} \circ \ldots \xi_{R^{m_j+i-1}f}$$

making $j \to \infty$ with i fixed, we obtain:

$$\bar{\xi}_0 \circ \bar{\xi}_1 \circ \ldots \circ \bar{\xi}_{i-1} R^i g = g^{2^i} \circ \bar{\xi}_0 \circ \ldots \circ \bar{\xi}_{i-1}$$

Define $D_{i,0} = \bar{\xi}_0 \circ \ldots \circ \bar{\xi}_{i-1}(B_0)$. It is invariant by g^{2^i} because

$$g^{2^i}(D_{i,0}) = g^{2^i}\bar{\xi}_0 \circ \ldots \circ \bar{\xi}_{i-1}(B_0) = \bar{\xi}_0 \circ \ldots \circ \bar{\xi}_{i-1} \circ R^i g(B_0) \subset D_{i,0}$$

Take

$$D_{i,k} = g^{l_0} \circ \bar{\xi}_0 \circ (Rg)^{l_1} \circ \bar{\xi}_1 \circ \ldots \circ (R^{i-2}g)^{l_{i-2}} \circ \bar{\xi}_{i-2} \circ (R^{i-1}g)^{l_{i-1}} \circ \bar{\xi}_{i-1}(B_0)$$

where $l_{i-1} \ldots l_2 l_1 l_0$ is the binary writing of the index $k.(0 \le k \le 2^i - 1)$

It is easy to check that $g^{2^i}(D_{i,k}) \subset D_{i,k}$. In fact $g(D_{i,k}) = D_{i,k+1}$ for $0 \le k \le 2^i - 2$, and $g(D_{i,2^i-1}) \subset D_{i,0}$.

Also, $\max_k \operatorname{diam}(D_{i,k}) \le \beta^i(\text{constant}) \to 0$ with $i \to \infty$.

From the definition of $D_{i,k}$, considering that $(R^i g)^{l_i}\bar{\xi}_i(B_0) \subset B_0$, we obtain that:

$$D_{i,k(\mathrm{mod}2^i)} \supset D_{i+1,k}$$

Let p_0^m be the fixed point of $R^m f$. Then

$$p_k^m = \xi_{R^m f} \circ \ldots \circ \xi_{R^{m+k-1}f} p_0^{m+k}$$

is a periodic point of period 2^k of $R^m f$.

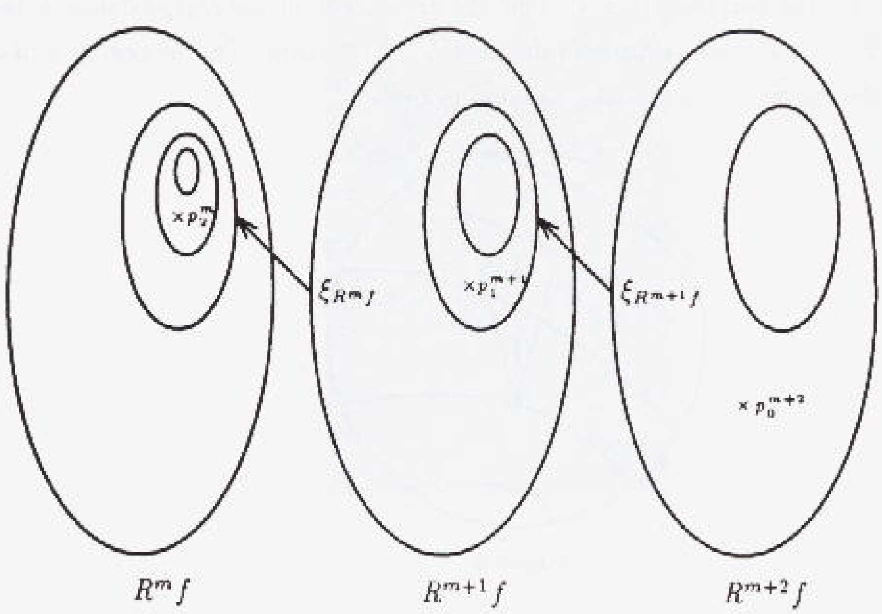


Figure 4

Again, there exist succesive subsequences of $\{m_j\}$ obtained making at each step k, $p_0^{m_j+k}$ convergent with k fixed and $j \to \infty$. As before, taking the diagonal subsequence, we have $p_k^{m_j} \to \bar{p}_k \in D_{k,0}$ where $\bar{p}_k$ is a periodic point of period 2^k of g. The orbit by g of $\bar{p}_k$ has one point on each $D_{k,j}$ $j = 0, \ldots, 2^k - 1$, because $g(D_{k,j}) \subset D_{k,j+1(mod 2^k)}$.

$Dg^{2^k}(\bar{p}_k) = \lim_{j\to\infty} D(R^{m_j} f)^{2^k}(p_k^{m_j})$ has an eigenvalue ρ_k smaller or equal than -1, because of the continuity of the eigenvalues. As $\det Dg = 0$, the other eigenvalue is zero.

For fixed $k \geq 0$, let us take $u_k = \lim_{j\to\infty} u_k(m_j)$ where $u_k(m_j)$ is the unitary expansive eigenvector for $D(R^{m_j}(f))^{2^k}(p_k^{m_j})$ and m_j is a subsequence such that $u_k(m_j)$ is convergent.

We have $Dg^{2^k}(\bar{p}_k)u_k = \rho_k u_k$, with $-1 \geq \rho_k > -\infty$. As

$$Dg^{2^k}(\bar{p}_k)u_k = \prod_{i=0}^{2^k-1} Dg(g^i(\bar{p}_k))u_k$$

we have that $\dim(\ker Dg(g^i(\bar{p}_k))) = 1$ for all $k \geq 0$, and all $i = 0, 1, \ldots, 2^k - 1$ and also we obtain that $\|Dg(g^{i(k)}(p_k))v_k\| \geq 1$ for some $i(k)$, and some unitary vector v_k.

Let us call $q_k = g^{i(k)}(\bar{p}_k)$. We have a sequence $\{(q_k, v_k)\}_{k\geq 1}$, with $q_k \in B_0$, $\|v_k\| = 1$, $\|Dg(q_k)v_k\| > 1$. Let us take now a subsequence k_j such that (q_{k_j}, v_{k_j}) is convergent to (q, v). We have $\|Dg(p)u\| > \frac{1}{2}$ for (p, u) in a neighborhood of (q, v). This neighborhood is an open set $V \subset B_0$ and a cone of unitary vectors that are not contracted more than $\frac{1}{2}$ by $Dg(p)$ as in the figure 5.

Thus, for all $p \in V$, $\dim \ker(Dg(p)) = 1$. The unitary vector of $\ker Dg(p)$ define a vectorfield in V of class C^{r-2}. For $r \geq 3$, this vectorfield define a C^{r-2} foliation. The image by g of each leaf is a point, because the derivative of g along the leaf is zero.

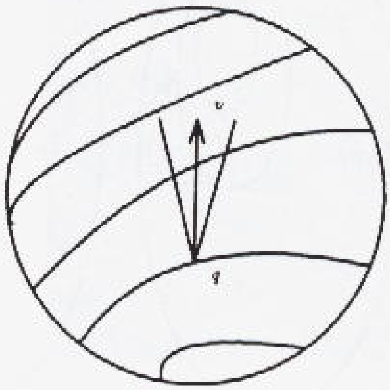


Figure 5

Take a set $D_{k_0 j_0}$ in V with k_0 sufficiently large so it is contained in a trivializing neighborhood of the foliation. Let us call D to the union of the leaves intersecting $D_{k_0 j_0}$. It is invariant by $g^{2^{k_0}}$. D has non void interior because $D_{k_0 j_0}$ is connected and has points of period 2^k for all $k \geq k_0$, that can not be contained in the same leaf of the foliation. Take $k = k_0 + 1$. In the trivializing coordinates (x, y) in D, each leaf corresponds to constant x. As the image by g (and any of its iterates) of each leaf is a point, we have for (x, y) in D: $g^{2^k}(x, y) = (g_1(x), g_2(x))$. Let us see that g_1 has at least one critical point.

We have in D a fixed point $q_k = (x_k, y_k)$ of g^{2^k}, and a fixed point $q_{k-1} = (x_{k-1}, y_{k-1})$ of $g^{2^{k-1}}$. As $Dg^{2^k}(q_{k-1})$ has a eigenvalue greater than one, it is obtained that $g_1'(x_k) > 1$. But, as $Dg^{2^k}(q_k)$ has a negative eigenvalue smaller that -1, we have that $g_1'(x_{k-1}) < -1$. There must exist at least one point where $g_1' = 0$. $\square$

The last theorem asserts that the C^r-cascades of period doubling in dimension 2 verifying the hypothesis of the uniform bounds can be studied as a perturbation of an one dimensional multimodal map.

In the section 3 we study a n-dimensional $(n \geq 2)$ example of Gambaudo and Tresser that is not reductible to dimension one, and in the section 4 we study the cascades of period doubling appearing when perturbing in n-dimensions the one-dimensional Feigenbaum's map. In both sections our purpose is to approximate the cascades with maps exhibiting homoclinic tangencies.

3 Approximation with homoclinic tangencies of the Gambaudo-Tresser n-dimensional cascade

The purpose of this section is to prove that the Gambaudo-Tresser [GT,1992] example of cascade of period doubling in dimension n is approximated with homoclinic tangencies. It is of type C^r with r increasing with n, and is not uniformly dissipative. Indeed at the points of the Cantor set the determinant of the jacobian matrix is one.

Theorem 3.1 (Gambaudo-Tresser, [GT,1992]) *For any $r > 1$ there exists $n \geq 2$ and a C^r-map of the n-dimensional ball that is a cascade of period doubling, whose Cantor set attractor contains an affine copy of itself scaled by a factor λ that can be chosen in an interval.*

Remark 3.2 As the geometry of the Cantor set can be chosen, this theorem implies that there is no hope of finding universal geometry of the Cantor set attractor. In other words this example can not be reductible to the Feigenbaum's one dimensional map.

The proof of the theorem is constructive. As we shall use later this construction, we include the proof of [GT,1992].

Proof: Let us define F_0, a C^∞ diffeomorphism in the unitary n-dimensional ball D verifying the following conditions:

a) F_0 is the identity in a thin shell $D \setminus D_{1-\gamma}$, where $D_{1-\gamma}$ is the ball of radius $1-\gamma$ concentric with D.

b) Consider 2^n disjoint balls $D_{1,i}$, $i = 0, \ldots, 2^n - 1$ of radius $\lambda < 1$, contained in $D_{1-\gamma}$ as in the figure 6, leaving enough room to move rigidly any pair of these disjoint balls until they exchange their positions. It is enough that $\lambda < \frac{1-\gamma}{2\sqrt{n}+1}$

There is an isotopy $\{\psi_t\}_{t\in[0,1]}$ from the identity $\psi_0 =$id map to $F_0 = \psi_1$; ψ_t restricted to $D_{1,i}$ for each $i = 0, \ldots 2^n - 1$ is a traslation, and $F_0(D_{1,i}) = D_{1,i+1(mod 2^n)}$.

c) F_0 has one single periodic orbit of period $1, 2, \ldots, 2^{n-1}$ of saddle type of stable codimension one in $M_1 = D_{1-\gamma} \setminus \cup_{i=0}^{2^n-1} D_{1,i}$, and no other periodic orbits in M_1.

d) The set $\cup_{i=0}^{2^n-1} D_{1,i}$ is an attractor for F_0, while the shell $D \setminus D_{1-\gamma}$ is an attractor for the inverse mapping F_0^{-1}.

We then have that $F_0^{2^n}|_{D_{1,i}}$ is the identity.

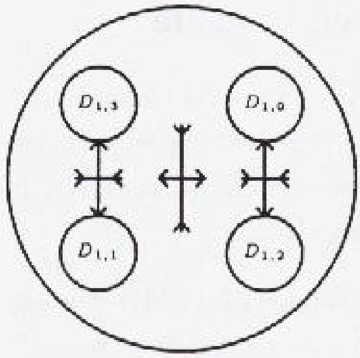


Figure 6

Let us modify F_0 in $\cup_{i=0}^{2^n-1} D_{1,i}$ by self similarity to obtain F_1 such that the behavior of $F_1^{2^n}|_{D_{1,0}}$ copies that of F_0 in D. Let F_1 be defined as F_0 in $D \setminus \cup_i D_{1,i}$ and

$$F_1|_{D_{1,i}} = \Lambda_{1,i+1(mod 2^n)} \circ \psi_{\frac{i+1}{2^n}} \circ \psi^{-1}_{\frac{i}{2^n}} \circ \Lambda^{-1}_{1,i}$$

where $\Lambda_{1,i}$ is the homotecy transforming the ball D onto $D_{1,i}$ (the homotecy rate is λ). F_1 is of class C^∞ because on each ball $D_{1,i}$, F_0 and F_1 coincide in the shell $D_{1,i} \setminus \Lambda_{1,i}(D_{1-\gamma})$, $F_1^{2^n}(D_{1,i}) = D_{1,i}$ and $F_1^{2^n}|_{D_{1,0}} = \Lambda_{1,0} F_0 \Lambda^{-1}_{1,0}$ because

$$F_1|_{D_{1,2^n-1}} \circ \ldots \circ F_1|_{D_{1,1}} \circ F_1|_{D_{1,0}} = \Lambda_{1,0} \circ \psi_1 \circ \Lambda^{-1}_{1,0} = \Lambda_{1,0} \circ F_0 \circ \Lambda^{-1}_{1,0}$$

For $i = 0, \ldots, 2^n - 1$, consider $\Lambda_{1,0}(D_{1,i})$. They are 2^n balls inside $D_{1,0}$ that are moved by translations with F_1 and its iterates, generating a family of 2^{2n} balls $D_{2,j}$, $j = 0, \ldots, 2^{2n} - 1$ of radius λ^2 inside the balls $D_{1,i}$ for $i = 0, \ldots, 2^n - 1$. Now: $F_1(D_{2,j}) = D_{2,j+1(mod 2^{2n})}$ for $j = 0, \ldots, 2^{2n} - 1$ and $F_1^{2^{2n}}|_{D_{2,j}} = \mathrm{id}$.

By induction, in the step $h \geq 1$ we modify F_{h-1} inside the 2^{hn} balls $D_{h,j}$, $j = 0, \ldots, 2^{hn} - 1$ of radius λ^h. Having $F_{h-1}^{2^{hn}}|_{D_{h,j}} = \mathrm{id}$, we construct F_h defined as follows:

$$F_h = F_{h-1} \text{ in } D \setminus \cup_j D_{h,j} \text{ and}$$

$$F_h|_{D_{h,j}} = \Lambda_{h,j+1(mod 2^{hn})} \circ \psi_{\frac{j+1}{2^{hn}}} \circ \psi^{-1}_{\frac{j}{2^{hn}}} \circ \Lambda^{-1}_{h,j} \tag{1}$$

where $\Lambda_{h,j}$ is the homotecy transforming the ball D onto $D_{h,j}$ (it is a homotecy of rate λ^h).

In this way we have defined in D a sequence of C^∞ maps $\{F_h\}_{h \geq 0}$. We claim that F_h is a Cauchy sequence in the topology C^r for certain r depending on n. So it defines a map F in the ball D, fixed by the renormalization $F = \Lambda^{-1}_{1,0} \circ F^{2^n} \circ \Lambda_{1,0}$.

In fact $F_h - F_{h-1}$ is null in the complement of $\cup_{j=0}^{2^{hn}-1} D_{h,j}$ so they differ only in the 2^{hn} balls of radius λ^h that are interchanged both with F_h and F_{h-1}. Thus $\|F_h - F_{h-1}\|_{C^0} < 2\lambda^h$ with $\lambda < 1$.

Now, the derivatives of $F_{h-1}|_{\cup_j D_{h,j}}$ are the identity because it is a traslation restricted to each of the balls $D_{h,j}$. It is left to prove for h large enough that

$$\|(DF_h - \mathrm{id})|_{\cup_j D_{h,j}}\|_{C^{r-1}} < k\alpha^h$$

with some $\alpha < 1$.

In fact, from (1)

$$\|DF_h - \mathrm{id}\|_{C^{r-1}} \le (\lambda^{-h})^{r-1}\|D(\psi_{\frac{j+1}{2^{hn}}}\psi^{-1}_{\frac{j}{2^{hn}}}) - \mathrm{id}\|_{C^{r-1}}$$

For the isotopy ψ_t we have that

$$\|\psi_t \circ \psi_s^{-1} - \mathrm{id}\|_{C^r} < k|t-s|$$

for all t and s such that $t-s$ is small enough. So

$$\|D(\psi_{\frac{j+1}{2^{hn}}}\psi^{-1}_{\frac{j}{2^{hn}}}) - \mathrm{id}\|_{C^{r-1}} \le k\frac{1}{2^{hn}}$$

We thus have

$$\|DF_h - \mathrm{id}\|_{C^{r-1}} \le k\left(\frac{1}{2^n\lambda^{r-1}}\right)^h$$

To have $\{F_h\}_{h\ge1}$ a Cauchy sequence it is enough that $2^n\lambda^{r-1} > 1$, that is, $n > -\frac{(r-1)\log\lambda}{\log 2}$. The interval in which λ can be chosen is $\frac{1-\gamma}{2\sqrt{n}+1} > \lambda > \frac{1}{2^{n/(r-1)}}$ for r, γ, n such that $2^{n/(r-1)} > \frac{2\sqrt{n}+1}{1-\gamma}$.

Now we have defined $F = \lim_{h\to\infty} F_h$ in the C^r topology. It is not still a cascade because it has a shell $D \setminus D_{1-\gamma}$ of fixed points and because of the self-similar construction it has shells inside the balls of generation h all formed by periodic points of period 2^h.

It is enough to change F_0 (and the isotopy ψ_t correspondingly) in a neighborhood of the shell $D \setminus D_{1-\gamma}$ so that D is mapped inside itself, and in a neighborhood of $\cup_{i=0}^{2^n-1} D_{1,i}$ so that the image of $D \setminus \cup_i D_{1,i}$ is not contained in itself. □

Theorem 3.3 *Let F be the C^r cascade of period doubling in dimension n of the theorem of Gambaudo-Tresser above. Given $\epsilon > 0$ there exists G of type C^r, exhibiting a homoclinic tangency and such that $\|G - F\|_{C^r} < \epsilon$.*

Proof: Let $\{\psi_t\}_{0\le t\le 1}$ be the isotopy such that $\psi_0 = \mathrm{id}$, $\psi_1 = F_0$ as in the proof of the last theorem. Define $\{\tilde{\psi}_t\}_{0\le t\le 1}$ such that $\tilde{\psi}_t = \psi_{2t}$ for $0 \le t \le \frac{1}{2}$, and $\{\tilde{\psi}_t\}_{\frac{1}{2}\le t\le 1}$ is the transformation $\delta_t \circ F_0$ where $\delta_{\frac{1}{2}}$ is the identity, and δ_t is constructed below.

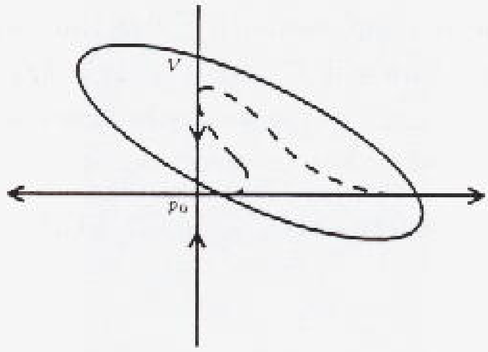


Figure 7

Let V be a connected open set, disjoint with $\cup_i D_{1,i}$, that does not contain any periodic point of F_0 and such that $W^s(p_0) \cap V$ and $W^u(p_0) \cap V$ are contained in fundamental domains of $W^s(p_0)$ and $W^u(p_0)$ respectively.

Now δ_1 is a map that is the identity in the complement of V and takes the points of the arc $W^u(p_0) \cap V$ and pushes them to be tangent to $W^s(p_0)$. This can be done with an isotopy $\{\delta_t\}_{\frac{1}{2} \leq t \leq 1}$ with $\delta_{\frac{1}{2}}$ =id, leaving fixed all the points of the complement of V. Now consider as in the proof of the last theorem the map F constructed as $\lim_{h \to \infty} F_h$ with F_h and F_{h+1} differing only in the balls $D_{h,j}$ of generation h. Let us define $\tilde{F}_h$ as follows:

$$\tilde{F}_h = \begin{cases} F & \text{in } D \setminus \cup_j D_{h,j} \\ \Lambda_{h,j+1(mod 2^{hn})} \circ \tilde{\psi}_{\frac{j+1}{2^{hn}}} \circ \tilde{\psi}^{-1}_{\frac{j}{2^{hn}}} \circ \Lambda^{-1}_{h,j} & \text{in } D_{h,j} \end{cases}$$

where $\Lambda_{h,j}$ is the homotecy transforming the ball D onto $D_{h,j}$. Now, by construction $\tilde{F}_h$ has a periodic point in $D_{h,0}$ of period 2^{hn} exhibiting a homoclinic tangency. It is left to show that $\|\tilde{F}_h - F\|_{C^r} \to_{h \to \infty} 0$.

$$\|\tilde{F}_h - F\|_{C^r} \leq \|\tilde{F}_h - F_h\|_{C^r} + \|F_h - F\|_{C^r}$$

As $F = \lim_{h \to \infty} F_h$, the second term of the sum above is less that ϵ for h large enough. As $F_h = F = \tilde{F}_h$ in $D \setminus \cup_j D_{h,j}$, and they differ in the balls $D_{h,j}$ that are interchanged both with F_h and $\tilde{F}_h$, we have

$$\|\tilde{F}_h - F_h\|_{C^0} < 2\,\mathrm{diam} D_{h,j} = 2\lambda^h \to_{h \to 0} 0$$

Now:

$$\|D(\tilde{F}_h - F_h)\|_{C^{r-1}} = \|D\tilde{F}_h - \mathrm{id}\|_{C^{r-1}} + \|DF_h - \mathrm{id}\|_{C^{r-1}} \leq$$

$$\leq \frac{1}{(\lambda^h)^{r-1}} \|D(\tilde{\psi}_{\frac{j+1}{2^{hn}}} \circ \tilde{\psi}^{-1}_{\frac{j}{2^{hn}}}) - \mathrm{id}\|_{C^{r-1}} + k\left(\frac{1}{2^n \lambda^{r-1}}\right)^h$$

For the isotopy $\tilde{\psi}$ we have

$$\|\tilde{\psi}_t \circ \tilde{\psi}_s^{-1} - \mathrm{id}\|_{C^r} < \tilde{k}|t-s| \text{ for all } t \text{ and } s$$

so

$$\|D(\tilde{F}_h - F_h)\|_{C^{r-1}} \leq \frac{k+\tilde{k}}{2^{hn}} \frac{1}{(\lambda^h)^{r-1}} = (k+\tilde{k})\left(\frac{1}{2^n\lambda^{r-1}}\right)^h \to_{h\to\infty} 0$$

Thus $\|\tilde{F}_h - F\|_{C^r} \to_{h\to\infty} 0$ as wanted. □

4 The analytic perturbations of the Feigenbaum's map

In this section we study the analytic cascades of period doubling appearing when perturbing in n dimensions the Feigenbaum's map. We show that the cascades are approximated with homoclinic tangencies. We develop the theory in the analytic case, exploting the fact that the renormalization is differentiable with derivative that is a compact operator whose spectrum is computable.

The main theorem to be proved in this section is

Theorem 4.1 *In the space $\mathcal{H}_D$ of n-dimensional, bounded and real analytic maps, there is a codimension one manifold $\mathcal{W}$, passing through the Feigenbaum's map Φ, such that any differentiable curve $\{G_\mu\}$ in $\mathcal{H}_D$, that intersects transversally $\mathcal{W}$ at G_{μ_∞}, verifies:*
a) It has period doubling bifurcations for a monotone sequence of parameter values $\mu_n \to \mu_\infty$.
b) There exists $\overline{\mu}_n \to \mu_\infty$ (monotonely, at the other side of μ_∞ than μ_n), such that $G_{\overline{\mu}_n}$ exhibits a homoclinic tangency.

4.1 Spectral analysis of the renormalization

Let us state some results in dimension one that give an understanding of the cascades of period doubling bifurcations.

Let D be a neighborhood of $[-1,1]$ in C, and $\mathcal{H}_D$ the space of real analytic maps defined and bounded in D. It is a Banach space with the supremum norm. In $\mathcal{H}_D$ let $\hat{M}$ be the manifold of even maps taking value 1 at 0:

$$\hat{M} = \{\psi \in \mathcal{H}_D : \psi(z) = g(z^2) \text{ for some } g \text{ real analytic, } g' \neq 0,\ g(0) = 1\}.$$

The renormalization transformation $\hat{\mathcal{F}}$ is defined as:

$$(\hat{\mathcal{F}}\psi)(x) = \psi(1)^{-1}\psi \circ \psi(\psi(1)x)$$

applied to the maps $\psi \in \hat{M}$ such that $-1 < \psi(1) < 0$ and $\psi(\psi(1)D) \subset D$.

The following theorem provides some properties of $\hat{\mathcal{F}}$:

Theorem 4.2 *If the neighborhood D is small enough, then:*

a) *There exists $\varphi \in \hat{M}$ fixed by $\hat{\mathcal{F}}$. When restricted to real arguments, $x\varphi'(x) < 0$ if $x \neq 0$, and the Schwartzian derivative $S\varphi$ is negative. Besides $\varphi^2(0) = \varphi(1) = \lambda = -0.3995\ldots$ and $\varphi''(0) = -1.52763\ldots$*

b) $\hat{\mathcal{F}}$ *is a* C^∞ *transformation, and* $d\hat{\mathcal{F}}(\varphi)$ *is a compact operator having a single eigenvalue* $\delta = 4.6692\ldots$ *of modulus greater or equal than 1, that is simple.*

c) The unstable manifold $\hat{W}^u(\varphi) \subset \hat{M}$ *intersects transversally the codimension-one manifold* $\hat{\Sigma}_1$ *of period doubling bifurcations, defined as follows:*

$$\hat{\Sigma}_1 = \{\psi \in \hat{M} : \psi'(x_0) = -1 \text{ for some } x_0 \text{ fixed by } \psi\}.$$

Proof: See O. Lanford III's article [L,1982]. Also [CE,1981] [VK,1982] [L,1984] [CER,1982] [VSK,1984]. This theorem was conjectured in [F,1978] [CT,1978] [F,1979].

Definition 4.3 The *Feigenbaum's map in dimension one,* is the map φ of the theorem above. The number $\delta = 4.6692\ldots$ is *the Feigenbaum's constant.*

Following Collet, Eckmann and Koch [CEK,1981], let us take a neighborhood D in C^n of the interval $[-1,1] \times \{0\}$(We denote $0 \in C^{n-1}$). Our functional space $\mathcal{H}_D$ will be the Banach real space formed by the real analytic maps defined and bounded in D, with the supremum norm.

Usually we will consider only the restrictions to R^n of the maps in $\mathcal{H}_D$(For simplicity we will not use a different notation to refer to the restriction).

Let us fix $\alpha \in R^{n-1}$, $\alpha \neq 0$, and define $\theta : C^n \mapsto C$, and $\theta_0 : C \mapsto C$, as follows:

$$\begin{aligned} \theta(z_0, Z) &= z_0^2 - \alpha \cdot Z \\ \theta_0(z) &= z^2 \end{aligned}$$

Definition 4.4 *The Feigenbaum's map in n dimensions* is the map:

$$\Phi = (f \circ \theta, 0) : D \subset C^n \mapsto C^n,$$

where $f \circ \theta_0 = \varphi$ is the Feigenbaum's map in dimension one, defined in 4.3.

For fixed α, there exists D small enough such that $\theta(D)$ is contained in the domain of f, and therefore Φ is well defined.

Being $\lambda = \varphi(1) = -0.3995\ldots \in (-1,0)$, let us define $\Lambda : C^n \mapsto C^n$, the linear rescaling $\Lambda(z_0, Z) = (\lambda z_0, \lambda^2 Z)$, and a (first) *renormalization transformation* :

$$\mathcal{N}G = \Lambda^{-1} \circ G \circ G \circ \Lambda$$

for all $G \in \mathcal{H}_D$ in a neighborhood of Φ.

The renormalization transformation $\mathcal{N}$ will be modified later (substituting the linear rescaling Λ with a nonlinear change of coordinates), to get a new renormalization transformation T that will have some desired properties. Observe that the Feigenbaum's map Φ is fixed by $\mathcal{N}$ and $d\mathcal{N}(\Phi)u = \Lambda^{-1} \circ (u \circ \Phi + d\Phi \circ \Phi \cdot u) \circ \Lambda$

Remark 4.5 In the sequel we will use the following denotation:

$$\Psi_\sigma = -\sigma \circ \Phi + d\Phi \cdot \sigma$$

for any given σ *analytic* : $C^n \mapsto C^n$

Ψ_σ is a map in $\mathcal{H}_D$ tangent at Φ to the curve of maps:

$$\{(I + t\sigma)^{-1} \circ \Phi \circ (I + t\sigma)\}, t \in (-\epsilon, \epsilon) \subset R,$$

of analytic conjugates of Φ near Φ. The eigenvectors of $d\mathcal{N}(\Phi)$ generated by such maps Ψ_σ are unessential.

Theorem 4.6 (Collet, Eckmann and Koch) [CEK,1981]

a) The map Φ (as defined in 4.4) is a fixed point of the renormalization $\mathcal{N}$.

b) $\mathcal{N}$ is infinitely differentiable and $d\mathcal{N}(\Phi)$ is a compact operator whose eigenvalues of modulus greater or equal than 1 are:

$$1, \lambda^{-1}, \lambda^{-2}, \delta$$

(where $\lambda = -0.3995\ldots$ and $\delta = 4.6692\ldots$).

c) Their respective spectral invariant subspaces S_0, S_1, S_2, and U, are eigenspaces. Besides, the subspace U is one-dimensional in $\mathcal{H}_D$ and the sum $S = S_0 \oplus S_1 \oplus S_2$ is the (n^2+n)-dimensional subspace of all the maps of the form Ψ_σ, where

$$\sigma(z_0, Z) = (a_0 + a_1 z_0,\ B_0 + B_1 z_0 + B_2 z_0^2 + A \cdot Z)$$

for some a_0 and a_1 in R; B_0, B_1 and B_2 in R^{n-1}; and $A \in \mathcal{L}(R^{n-1}, R^{n-1})$.

Proof: See [CEK,1981].

Many renormalizations for n-dimensional maps can be used. We have chosen one of them, obtained from the transformation $\mathcal{N}$ that fits well with the renormalization in dimension one, and with the theorem of Eckmann and Wittwer [EW,1987].

Definition 4.7 As an intermediate step, let us define *the transformation $\mathcal{F}$*, applied to the maps $G = (g_0, g) \in \mathcal{H}_D$ in a neighborhood of Φ:

$$\mathcal{F}(G) = \Lambda_G^{-1} \circ G \circ G \circ \Lambda_G,$$

where $\Lambda_G(z_0, Z) = (\lambda_G z_0,\ \lambda_G^2 Z)$ for

$$\lambda_G = \frac{g_0 \circ G(0,0)}{g_0(0,0)}$$

.

Note that $\lambda_\Phi = \lambda$ and $\mathcal{F}(\Phi) = \mathcal{N}(\Phi) = \Phi$. Besides, for all $u = (u_0, U) \in \mathcal{H}_D$:

$$d\mathcal{F}(\Phi)u = d\mathcal{N}(\Phi)u + a(u)\Psi_{\sigma_1}, \tag{2}$$

where:

$$a(u) = \frac{u_0(1,0)}{\lambda} + u_0(0,0)\left(\frac{1}{\lambda^2} - 1\right) - \frac{\alpha \cdot U(0,0)}{2\lambda}$$

and $\sigma_1(z_0, Z) = (z_0, 2Z)$. Using 4.5, we have:

$$\Psi_{\sigma_1}(z_0, Z) = (-f(z_0^2 - \alpha \cdot Z) + 2f'(z_0^2 - \alpha \cdot Z)(z_0^2 - \alpha \cdot Z),\ \ 0\ ).$$

A consequence of (2) and of the theorem 4.6, is the following:

Proposition 4.8 .

a) *Φ is a fixed point of $\mathcal{F}$, and $d\mathcal{F}(\Phi)$ has the same spectrum that $d\mathcal{N}(\Phi)$.*

b) *The sum of the spectral invariant subspaces corresponding to the eigenvalues* 1, λ^{-1} *and* λ^{-2} *is* $S = S_0 + S_1 + S_2$ (as in the theorem 4.6).

c) *For any $u \in \mathcal{H}_D$ there exists $\sigma[u]$, the unique analytic map in C^n such that $\Psi_{\sigma[u]} = Eu$, where E is the spectral projection on S. The transformation $u \mapsto \sigma[u]$ is linear and bounded.*

Proof: Let us denote $F = d\mathcal{F}(\Phi)$, $N = d\mathcal{N}(\Phi)$. They are compact operators. Denote $\Sigma(F)$, $\Sigma(N)$ their spectra. Theorem 4.6 and (2) above imply $F\Psi_{\sigma_1} = N\Psi_{\sigma_1} = \Psi_{\sigma_1}$. Let $\mu \neq 0$. We assert that $\mu \in \Sigma(F)$ with multiplicity m, if and only if $\mu \in \Sigma(N)$ with the same multiplicity. In fact, take $\mu \in \Sigma(F)$, with spectral subspace $\ker(F-\mu)^\nu$ of dimension m. Define $V = \ker(F-\mu)^\nu + [\Psi_{\sigma_1}]$. It is invariant by F. The Jordan matrix J of F restricted to V has μ in the diagonal repeated m times (and a single 1 if $\mu \neq 1$). In the same basis, the linear operator N restricted to V has a triangular matrix with the same diagonal than J, (due to (2)). Then

$\mu \in \Sigma(N)$ and has multiplicity at least m. Changing the roles of N and F, our assertion is proved, and also for $\mu = 1, \lambda^{-1}$ or λ^{-2}:

$$\ker(F-\mu)^{\nu} + [\Psi_{\sigma_1}] = \ker(N-\mu)^{\nu} + [\Psi_{\sigma_1}].$$

Now, part *b)* follows easily. Finally, $Eu = \Psi_{\sigma}$ for some σ in the set

$$\{\sigma : C^n \mapsto C^n \text{analytic}; \ \sigma(z_0, Z) = (a_0 + a_1 z_0, \ B_0 + B_1 z_0 + B_2 z_0^2 + A \cdot Z)\}$$

Call $Q : \sigma \mapsto \Psi_{\sigma}$ the linear transformation between finite-dimensional spaces. It is easy to check that Q is injective. Therefore $u \mapsto \sigma[u] = Q^{-1}Eu$ is linear and bounded. $\square$

We are now ready to define our final renormalization transformation in n dimensions:

Definition 4.9 The *renormalization transformation* T is:

$$T(G) = (I - \sigma[\mathcal{F}(G) - \Phi])^{-1} \circ \mathcal{F}(G) \circ (I - \sigma[\mathcal{F}(G) - \Phi]),$$

applied to $G \in \mathcal{H}_D$ in a neighborhood of Φ.

The renormalization T was chosen so that it verifies the following properties:

Corollary 4.10 (of the theorem 4.6) .

a) The map Φ is a fixed point of T.

b) T is infinitely differentiable and $dT(\Phi)$ is a compact operator, having a single simple eigenvalue $\delta = 4.6692\ldots$ of modulus greater or equal than 1.

c) The unstable manifold $\mathcal{W}^u(\Phi) = \{\Phi_\mu\}$ is formed by the maps $\Phi_\mu \in \mathcal{H}_D$ of the form:

$$\Phi_\mu(z_0, Z) = (f_\mu(z_0^2 - \alpha \cdot Z), \mathbf{0}),$$

where $f_\mu(z^2) = \varphi_\mu(z)$ are the one-dimensional maps of the unstable manifold $\{\varphi_\mu\} = \hat{\mathcal{W}}^u(\varphi)$ of the renormalization $\hat{\mathcal{F}}$ in dimension one. (cf. theorem 4.2)

Proof: Part *a)* can be easily verified. We begin showing *c)*. Let $\hat{\mathcal{M}}$ be the manifold of one-dimensional maps ψ that are real analytic, bounded, even and with the condition $\psi(0) = 1$.

We will say that Ψ is a *one-dimensional endomorphism* if it belongs to the following set:

$$M = \{\Psi \in \mathcal{H}_D : \Psi = (g \circ \theta, \mathbf{0}) \text{ with } g \text{ real analytic }, g' \neq 0, \ g(0) = 1\}$$

(Recall that $\theta(z_0, Z) = z_0^2 - \alpha \cdot Z$.)

The theorem 4.2 states that $\mathcal{F}$ restricted to M has a single eigenvalue δ of modulus greater or equal than 1 at the fixed point Φ. In other words, all the vectors in $T_\Phi M$ belong to $\ker(E)$ (see proposition 4.8). Thus $\sigma[u] = 0$ for all $u \in T_\Phi M$, and due to the definition 4.9, we have that *the affine manifold M of one-dimensional endomorphisms in $\mathcal{H}_D$ is invariant by the renormalization T.*

Part c) of the theorem is a consequence of *b)* and of the above remark. Finally, part *b)* follows from the proposition 4.8: in fact, taking derivatives in the equality of the definition 4.9, and denoting $F = d\mathcal{F}(\Phi)$, we get:

$$\begin{aligned} dT(\Phi)u &= Fu + \sigma[Fu] \circ \Phi - d\Phi \cdot \sigma[Fu] = \\ &= Fu - \Psi_{\sigma[Fu]} = (I - E)Fu. \end{aligned}$$

Now, all vectors of S are in the kernel of $dT(\Phi)$. Thus, the only unstable direction that remains has eigenvalue δ, as wanted. □

4.2 Homoclinic bifurcating maps

We will work with a particular type of homoclinic bifurcation, that is produced in compact parts of the stable and unstable manifolds. For the stable codimension-one case, the bifurcations are the unfolding of homoclinic tangencies.

Let $\{G_\mu\}$, $\mu \in [a, b]$, be a continuous arc in $\mathcal{H}_D$. Let us suppose that, for all $\mu \in [a, b]$, there exists a hyperbolic periodic point p_μ, depending continously on μ, of stable codimension one.

Let us denote A^u_μ and A^s_μ compact parts of $W^u(p_\mu)$ and $W^s(P_\mu)$ respectively, depending continously on μ, as C^1 submanifolds with boundary of R^n. (The point p_μ does not necessarily belong to A^u_μ or A^s_μ).

Definition 4.11 The arc $\{G_\mu\}$, $\mu \in [a, b]$ in $\mathcal{H}_D$ exhibits a *homoclinic bifurcation with unavoidable tangency* if there exist p_μ, A^u_μ, A^s_μ as above, such that:

i. *$\partial A^u_\mu \cap A^s_\mu = \partial A^s_\mu \cap A^u_\mu = \emptyset$, for all $\mu \in [a, b]$*

ii. *$A^u_a \cap A^s_a = \emptyset$*

iii. *$A^u_b \cap A^s_b$ contains at least one point of transversal intersection.*

The name *unavoidable tangency* of the definition above is due to the following:

Proposition 4.12 *If $\{G_\mu\}$, $\mu \in [a,b]$, is an arc as in the definition 4.11, then there exists $\mu_0 \in [a,b]$ such that G_{μ_0} has a periodic point with a homoclinic tangency.*

Proof: As the interval $[a,b]$ is connected, there exists $\mu_0 \in (a,b)$ such that A^s_μ and A^u_μ have a non transversal intersection. It must be a tangency because the dimension of A^u_μ is one. □

We will take the definition of *band-merging maps* from [EW,1987], and relate it with the homoclinic bifurcations.

Let $\psi \in \hat{M}$ be a one-dimensional map, restricted to real argument. (Recall that $\hat{M}$ is the set of even real analytic maps defined in the neighborhood D of $[-1,1]$ and such that $\psi(0) = 1$.)

Definition 4.13 A map $\psi \in \hat{M}$ is *band-merging* if:

i. $x\psi'(x) < 0$ *for all* $x \neq 0$, and

ii. $0 < \psi \circ \psi(1) = -\psi(1) < 1$

As $\psi(x) = g(x^2)$, we have the following equivalent definition: $g' < 0$; $-1 < g(1) < 0$; $g(g(1))^2 = -g(1)$.

Proposition 4.14 *If ψ is band-merging then:*

a) $\hat{\mathcal{F}}\psi(-1) = -1$ and $x(\hat{\mathcal{F}}\psi)'(x) < 0$ if $x \neq 0$.

b) Besides, if $S\psi < 0$ then:

b1) $-\psi(1)$ is a hyperbolic repellor, whose repelling basin includes $[\psi(1), -\psi(1)]$,

b2) any $\tilde{\psi} \in \hat{M}$, near enough ψ, has a repelling fixed point whose basin includes $[\tilde{\psi}(1), -\tilde{\psi}(1)]$.

Proof: Part *a)* is a straightforward verification. Let us see part *b)* :

The map $\psi \circ \psi$ is increasing in $(0, x_{-1})$, where $x_{-1} > 0$ and $\psi(x_{-1}) = 0$. Its graph, at $x = 0$ is below the diagonal, at x_{-1} is above the diagonal, and at $x_0 = -\psi(1)$ intersects the diagonal. By contradiction, suppose that $(\psi \circ \psi)'(x_0) \leq 1$. Then, there exists x_1 where $(\psi \circ \psi)''$ vanishes and $(\psi \circ \psi)'''$ is non negative. This implies that $S(\psi \circ \psi)(x_1) \geq 0$, contradicting our hypothesis. The same contradiction is obtained if $\psi \circ \psi$ is supposed to have other fixed point $\overline{x}_0 \in [0, x_0)$. Therefore x_0 is a repellor and $[0, x_0]$ is in its basin. By symmetry, also $[-x_0, 0]$ is. This proves *b1)*. To show part *b2)* , consider any $\tilde{\psi}$ near enough ψ, so that it also has a hyperbollic repellor, and $S\tilde{\psi} < 0$. The proof also works for $\tilde{\psi}$ instead of ψ. □

Remark 4.15 Due to the above proposition, the band merging maps with negative Schwartzian derivative satisfy the condition that the critical point lands after finitely many iterations on an unstable periodic point.

In the following theorem the family $\{\varphi_\mu\}$ is the unstable manifold in $\hat{M}$ of the hyperbolic fixed point φ of the renormalization $\hat{\mathcal{F}}$.

Theorem 4.16 (Eckmann and Wittwer) .
There exists μ_0 such that $\varphi_{\mu_0} \in \hat{\mathcal{W}}^u(\varphi)$ is band-merging, and for all μ near μ_0:

$$\frac{\partial}{\partial\mu}(\varphi_\mu(\varphi_\mu(1)) + \varphi_\mu(1)) \neq 0$$

Proof: See [EW,1987]

This last theorem asserts that $\hat{\mathcal{W}}^u(\Phi)$ intersects, transversally in $\hat{M}$, at φ_{μ_0}, the codimension one (in $\hat{M}$) manifold of band-merging maps.

The following lemma is a consequence of the theorem 4.16.

Lemma 4.17 *Given $\epsilon > 0$ there exists $\gamma > 0$ such that, for any interval $[a,b] \subset (\mu_0 - \gamma, \mu_0 + \gamma)$ containing μ_0 in its interior, the arc $\{\Phi_\mu\}$, $\mu \in [a,b]$ exhibits a homoclinic bifurcation with unavoidable tangency, and the first coordinate projection of the compact part A_μ^** (cf. definition 4.11 *) is contained in $(-\epsilon, \epsilon)$.*

Proof: First, we assert that $\Phi_{\mu_0} = (f_{\mu_0} \circ \theta, 0)$ has a hyperbolic fixed point $p_{\mu_0} = (-f_{\mu_0}(1), \mathbf{0})$ of stable codimension one. In fact, it is fixed because $f_{\mu_0} \circ \theta_0$ is band merging (cf. theorem 4.16). Let us see that it is hyperbolic, computing $D\Phi_{\mu_0}(p_{\mu_0})$:

$$D\Phi_{\mu_0} = \begin{bmatrix} 2xf'_{\mu_0} \circ \theta & (f'_{\mu_0} \circ \theta)\alpha \\ 0 & \mathbf{0} \end{bmatrix}$$

with $2xf'_{\mu_0}(x^2) = (f_{\mu_0} \circ \theta_0)'(x)$. But $f_{\mu_0} \circ \theta_0$ belongs to the unstable manifold $\hat{\mathcal{W}}^u(\varphi)$ in $\hat{M}$, and all maps in $\hat{\mathcal{W}}^u(\varphi)$ have negative schwarzian derivative (because all the maps in a neighborhood of φ have, and also their renormalizations). Therefore, proposition 4.14 states that $-f_{\mu_0}(1)$ is a repellor. Thus:

$$|2x_0 f'_{\mu_0}(x_0^2)| > 1 \text{ for } x_0 = -f_{\mu_0}(1)$$

Thus, our first assertion is proved.

Let us choose $\gamma > 0$ small enough so that, for all $\mu \in B_\gamma(\mu_0)$ there exists $p_\mu = (x_\mu, \mathbf{0})$, continuation of p_{μ_0}, hyperbolic fixed point of $\Phi_\mu = (f_\mu \circ \theta, \mathbf{0}) \in \mathcal{W}^u(\Phi)$. Here x_μ is the hyperbolic repellor of the unimodal map $\varphi_\mu = f_\mu \circ \theta_0$, whose repelling basin includes $[f_\mu(1), -f_\mu(1)]$, as proved in the proposition 4.14.

We define:

$$A_\mu^u = \{(x_0, X) : X = 0, |x_0| \leq -f_\mu(1)\} \subset W^u(p_\mu).$$

The theorem 4.16 allows us to choose $[a,b] \subset B_\gamma(\mu_0)$ such that $\varphi_\mu(\varphi_\mu(1)) + \varphi_\mu(1)$ is positive for $\mu \in [a, \mu_0)$ and negative for $\mu \in (\mu_0, b]$. (We write $[a,b]$ although b could be smaller than a.)

We assert that, given $\delta > 0$ there exists $[a,b]$ sufficiently small and $y_\mu \in \varphi_\mu^{-2}(x_\mu)$, for all $\mu \in [a,b]$, such that:

$$\begin{array}{ll} 1 < y_\mu < 1+\delta & \text{if} \quad \mu \in [a,\mu_0) \\ y_{\mu_0} = 1 & \\ 1-\delta < y_\mu < 1 & \text{if} \quad \mu \in (\mu_0, b] \end{array}$$

In fact, if $\mu \in [a, \mu_0)$ we have $\varphi_\mu(\varphi_\mu(1)) + \varphi_\mu(1) > 0$, i.e. the graph of φ_μ at $-\varphi_\mu(1)$ is above the diagonal. As φ_μ is decreasing in $(0,1]$, the fixed point x_μ is at right of $-\varphi_\mu(1)$. Therefore, given $\delta_1 > 0$:

$$\varphi_\mu(1) > -x_\mu > \varphi_\mu(1) - \delta_1$$

for all $\mu \in [a, \mu_0)$, near enough μ_0.

The map φ_μ is decreasing at right of 0 and defined in a neighborhood D of $[-1,1]$. We conclude that, given $\delta > 0$, there exists a, near enough μ_0, and $y_\mu \in \varphi_\mu^{-1}(-x_\mu)$ for all $\mu \in [a, \mu_0)$, such that $1 < y_\mu < 1+\delta$. As $-x_\mu \in \varphi_\mu^{-1}(x_\mu)$, we have $y_\mu \in \varphi_\mu^{-2}(x_\mu)$. The same argument, with the opposite inequalities, is valid for $\mu \in (\mu_0, b]$. This completes the proof of our assertion.

We have $f_\mu(0) = 1$ and $f_\mu'(0) = 1/2\varphi_\mu''(0) < 0$ for all $\varphi_\mu \in \hat{W}^u(\varphi)$. For any $\mu \in [a,b]$, near μ_0, the map f_μ is invertible and decreasing in a fixed neighborhood of 0. Let us denote $\epsilon_\mu = f_\mu^{-1}(y_\mu)$. Our previous assertion can be reformulated as follows:

Given $\epsilon > 0$, there exists $[a,b]$ sufficiently small, such that:

$$\begin{array}{ll} -\epsilon^2 < \epsilon_\mu < 0 & \text{if} \quad \mu \in [a,\mu_0) \\ \epsilon_{\mu_0} = 0 & \\ 0 < \epsilon_\mu < \epsilon^2 & \text{if} \quad \mu \in (\mu_0, b] \end{array}$$

With no loss of generality, let us suppose that $\alpha_{n-1} \neq 0$. (Recall that $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{n-1}) \neq 0$.) Let us denote $X = (X_1, X_2, \ldots, X_{n-1})$. Now we can define, for given $\epsilon > 0$, :

$$A_\mu^s = \{(x_0, X) : x_0^2 - \alpha X = \epsilon_\mu, |x_0| \leq \epsilon, ||(X_1, \ldots, X_{n-2})|| \leq \epsilon\}.$$

It is easy to check that $A_\mu^s \subset \Phi_\mu^{-3}(p_\mu) \subset W^s(p_\mu)$.

Let us see how A^s_μ looks: For $\mu = \mu_0$, $\epsilon_{\mu_0} = 0$ and $A^s_{\mu_0} \subset \{x_0^2 - \alpha X = 0\}$. It is a quadratic codimension-one manifold of R^n, passing through (0,0) and tangent at (0,0) to $A^u_{\mu_0}$. For $\mu \in [a, \mu_0)$, $\epsilon_\mu < 0$, and A^s_μ does not intersect $\{X = \mathbf{0}\} \supset A^u_\mu$. For $\mu \in (\mu_0, b]$, $\epsilon_\mu \in (0, \epsilon^2)$. So A^s_μ intersects $\{X = \mathbf{0}\}$ at two points $q = (-\sqrt{\epsilon_\mu}, \mathbf{0})$ and $r = (\sqrt{\epsilon_\mu}, \mathbf{0})$, both in the ϵ-neighborhood of (0,0). Then both q and r are in A^u_μ.

Besides $T_q A^s_\mu$ and $T_r A^s_\mu$ are transversal to the subspace $\{X = \mathbf{0}\} = T_q A^u_\mu = T_r A^u_\mu$.

Finally, if ϵ is chosen small enough, we get $\partial A^s_\mu \cap A^u_\mu = \partial A^u_\mu \cap A^s_\mu = \emptyset$, for all $\mu \in [a, b]$. □

Now we are ready to perturb the family $\{\Phi_\mu\}$, $\mu \in [a, b]$, contained in $\mathcal{W}^u(\Phi)$, and prove that the homoclinic bifurcation persists for nearby families.

Lemma 4.18 *There exists an interval $[a, b]$ and neighborhoods N, N_1 and N_2 in $\mathcal{H}_D$, of $\{\Phi_\mu : \mu \in [a, b]\}$, Φ_a and Φ_b respectively, such that any continuous arc $\{G_\mu\}$, in N, with extremes in N_1 and N_2, exhibits a homoclinic bifurcation with unavoidable tangency.*

Proof: Let us take $\Phi_{\mu_0} = (f_{\mu_0} \circ \theta, \mathbf{0})$ with μ_0 as in the theorem 4.16 (i.e. $f_{\mu_0} \circ \theta_0$ is band merging). We have that $p_{\mu_0} = (-f_{\mu_0}(1), \mathbf{0})$ is a fixed point, of saddle type. Its local stable manifold is contained in $\{(x_0, X) : x_0^2 - \alpha X - (f_{\mu_0}(1))^2 = 0\}$. Any G in a small neighborhood of Φ_{μ_0} in $\mathcal{H}_D$, has an hyperbolic fixed point $p(G)$, whose local stable manifold is of codimension one, contained in

$$\{(x_0, X) : U(x_0, X, G) = 0\},$$

where $U(\cdot, \cdot, G)$ is a real function of (x_0, X), depending continously on G. [HPS,1977] [PM,1982] We have $U(x_0, X, \Phi_{\mu_0}) = x_0^2 - \alpha X - (f_{\mu_0}(1))^2$.

Let us define, for any (x_0, X, G) in a certain small neighborhood of $(0, \mathbf{0}, \Phi_{\mu_0})$ in $R^n \times \mathcal{H}_D$, the real function:

$$F(x_0, X, G) = U(G^3(x_0, X), G)$$

The point $(0, \mathbf{0})$ verifies $\Phi^3_{\mu_0}(0, \mathbf{0}) = p_{\mu_0}$, and so

$$F(0, 0, \Phi_{\mu_0}) = 0$$

As in the proof of the previous lemma, let us suppose $\alpha_{n-1} \neq 0$, and compute the partial derivative:

$$\frac{\partial F}{\partial X_{n-1}}(0, \mathbf{0}, \Phi_{\mu_0}) = 2\alpha_{n-1} f_{\mu_0}(1)(\varphi^2_{\mu_0})'(1) f'_{\mu_0}(0) \neq 0$$

Now, by the implicit function theorem, there exists N_0, neighborhood of Φ_{μ_0} in $\mathcal{H}_D$, and $\epsilon > 0$ such that for all $G \in N_0$, for all $x_0 \in B_\epsilon(0)$ and for all $(X_1, \ldots, X_{n-2})$ in R^{n-2} with norm less than ϵ, is defined the coordinate $X_{n-1} = u(x_0, X_1, \ldots, X_{n-2}, G)$ verifying:

$$(x_0, X) \in G^{-3}(W^s_{loc}p(G)) \subset W^s(p(G))$$

with $X = (X_1, \ldots, X_{n-1})$.

Let us take

$$A^s(G) = \{(x_0, X) : \; X_{n-1} = u(x_0, X_1, \ldots, X_{n-2}, G); \; |x_0| \leq \epsilon; \; ||(X_1, \ldots, X_{n-2})|| \leq \epsilon\}.$$

We have that $A^s(G)$ is a C^1 submanifold with boundary of R^n, that depends continously on $G \in N_0$. It is a compact part of $W^s(p(G))$. For the neighborhood N_0 and $\epsilon > 0$ as above, let us take $[a, b]$ and A^s_μ as in the previous lemma, and also such that $\Phi_\mu \in N_0 \; \forall \mu \in [a, b]$. We have that $A^s_\mu = A^s(\Phi_\mu)$.

On the other hand, the compact piece A^u_μ of unstable manifold is contained, for some fixed n_0 (independent of μ) in $\Phi^{n_0}_\mu(W^u_{loc}(p(\Phi_\mu))$. Let us take for any $\mu \in [a, b]$, a small neighborhood N_μ of Φ_μ in N_0, such that $G^{n_0}(W^u_{loc}(p(G))$ is as C^1 proximate to $\Phi^{n_0}_\mu(W^u_{loc}(p(\Phi_\mu))$ as needed, for any $G \in N_\mu$. Consequently, compact parts $A^u(G)$ and $A^s(G)$ can be chosen, as proximate as needed from A^u_μ and A^s_μ respectively (as C^1 submanifolds with boundary), for any $G \in N_\mu$. The three conditions in the definition 4.11 are persistent under small C^1 perturbations of A^u_μ and A^s_μ. Therefore, the lemma is proved taking

$$N = \bigcup_{\mu \in [a,b]} N_\mu \; ; \quad N_1 = N_a \; ; \quad N_2 = N_b$$

□

Now, we are ready to complete the proof of the theorem 4.1.

Proof: Let $\mathcal{W} = \mathcal{W}^s(\Phi)$. The lemmas above state the existence of the arc $\{\Phi_\mu : \mu \in [a, b]\} \subset \mathcal{W}^u(\Phi)$ and the neighborhoods N, N_1, N_2. Given a curve $\{G_\mu\}$, transversal at $\mu = \mu_\infty$ to $\mathcal{W}$, its images by the renormalization T^n accumulate, when $n \to \infty$, at the unstable manifold of Φ. (See [PM,1982]). In particular they approach the arc $\{\Phi_\mu : \mu \in [a, b]\} \subset \mathcal{W}^u(\Phi)$. Consequently, there exists $[a_n, b_n]$, for all n sufficiently large, such that $T^n G_{a_n} \in N_1$, $T^n G_{b_n} \in N_2$, $T^n G_\mu \in N$ for all $\mu \in [a_n, b_n]$.

Besides $[a_n, b_n] \to \mu_\infty$, because the argument above works for any subarc of $\{G_\mu\}$ as near as wanted from G_{μ_∞}.

The lemma 4.18 states that $\{T^n G_\mu\}$, $\mu \in [a_n, b_n]$ exhibits a homoclinic bifurcation with unavoidable tangency, and so there exists $\overline{\mu}_n \in [a_n, b_n] \to \mu_\infty$ with $G_{\overline{\mu}_n}$ exhibiting a homoclinic tangency. □

References

[BC,1991] M.Benedicks, L.Carleson: *The dynamics of the Hénon map.* Annals of Math. 133 (73–169) 1991

[BGLT,1993] C.Bonatti, J.M.Gambaudo, J.M.Lion, C.Tresser: *Wandering domains for infinitely renormalizable diffeomorphisms of the disk*Preprint Inst. Non Lineaire de Nice 1993

[CE,1981] M.Campanino, H.Epstein: *On the existence of Feigenbaum's fixed point.* Comm. Math. Phys. 79 (261–302) 1981

[CEK,1981] P.Collet, J.P.Eckmann, H.Koch: *Period doubling bifurcations for families of maps on* R^n. Journ. of Stat. Phys., 25, N.1 (1–14) 1981

[CER,1982] M.Campanino, H.Epstein, D.Ruelle: *On Feigenbaum's functional equation.* Topology 21 N.2 (125–129) 1982

[CT,1978] P.Coullet, C.Tresser: *Itérations d'endomorphisms et groupe de rénormalisation.* C.R. Acad.Sci.Paris 287 (577–588) 1978

[EW,1987] J.P.Eckmann, P.Wittwer: *A complete proof of the Feigenbaum conjectures.* Journ. of Stat. Physics 46, N.3/4 (455–475) 1987

[F,1978] M.J.Feigenbaum: *Quantitative universality for a class of nonlinear transformations.* Journ. Stat. Physics, vol 19 (25–52) 1978

[F,1979] M.J.Feigenbaum: *The universal metric properties of non-linear transformations.* Journ. Stat. Physics, vol 21 (669–706) 1979

[GT,1992] J.M.Gambaudo,C.Tresser: *Self similar constructions in smooth dynamics: rigidity, smothness and dimension.* Comm. in Math. Physics 150 (45–58) 1992

[HPS,1977] M.Hirsch, C.Pugh, M.Shub: *Invariant manifolds.* Lect. Notes in Math. 583 (Springer) 1977

[L,1982] O.Lanford III: *A computer assisted proof of the Feigenbaum's conjectures.* Bulletin of the A.M.S. 6 (427–434) 1982

[L,1984] O.Lanford III: *A shorter proof of the existence of the Feigenbaum fixed point.* Commun. Math. Phys. 96 (521–538) 1984

[M,1991] J.C.Martín-Rivas: *Homoclinic bifurcations and cascades of period doubling in higher dimensions.* Tese do IMPA (Rio de Janeiro) 1991

[MS,1993] W.de Melo, S.van Strien: *One-dimensional dynamics.* Springer-Verlag 1993

[MV,1992] L.Mora, M.Viana: *Abundance of Strange Attractors.* Acta Math. (to appear)

[N,1979] S.Newhouse: *The abundance of wild hyperbolic sets and nonsmooth stable sets for diffeomorphisms.* Publ. Math. I.H.E.S. 50 (101–151) 1979

[NPT,1983] S.Newhouse, J.Palis, F.Takens: *Bifurcations and Stability of Families of Diffeomorphisms.* Publ. Math. I.H.E.S. 57 (5–71) 1983

[PM,1982] J.Palis, W.de Melo: *Geometric Theory of Dynamical Systems. An Introduction.* Springer-Verlag (New York) 1982

[PT,1987] J.Palis, F.Takens: *Hyperbolicity and the creation of homoclinic orbits.* Annals of Math. 125 (337–374) 1987

[PT,1993] J.Palis, F. Takens: *Hyperbolicity and sensitive chaotic dynamics at homoclinic bifurcations.* Cambridge University Press 1993

[PV,1991] J.Palis, M.Viana: *High dimension diffeomorphisms displaying infinitely many sinks.* (to appear)

[R,1992] N.Romero: *Persistence of homoclinic tangencies in higher dimensions.* Tese do IMPA (Rio de Janeiro) 1992

[S,1987] M.Shub: *Global stability of Dynamical Systems.* Springer-Verlag (New York) 1987

[S,1991] D.Sullivan: *Bounds, quadratic differentials and renormalization conjectures.* Mathematics into the twenty-first century. Vol.2 AMS 1991

[T,1986] F.Takens: *Homoclinic Bifurcations.* Proc. Int. Congress of Math.,Berkeley (1229–1236) 1986

[V,1991] M.Viana: *Strange attractors in higher dimensions.* Tese do IMPA (Rio de Janeiro) 1991

[VK,1982] E.Vul, K.Khanin: *The unstable separatrix of Feigenbaum's fixed point.* Russ. Math. Surveys 37(5) (200–201) 1982

[VSK,1984] E.Vul, Ya.Sinai, K.Khanin: *Feigenbaum universality and the thermodinamic formalism.* Usp.Mat.Nauk 39:3 and Russ. Math. Surv. 39:3 (3–37 and 1–40 resp.) 1984

[YA,1983] J.A.Yorke, K.T.Alligood: *Cascades of period doubling bifurcations: a prerrequisite for horseshoes.* Bull.A.M.S. 9 (319–322) 1983