# Observable Optimal State Points of Subaddicive Potentials

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### Abstract

For a sequence of subadditive potentials, a method of choosing state points with negative growth rates for an ergodic dynamical system was given in [5]. This paper first generalizes this result to the non-ergodic dynamics, and then proves that under some mild additional hypothesis, one can choose points with negative growth rates from a positive Lebesgue measure set, even if the system does not preserve any measure that is absolutely continuous with respect to Lebesgue measure.

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## 1  Introduction

Let $f : M \to M$ be a continuous map on a compact, finite-dimensional manifold $M$, and $m$ a normalized Lebesgue measure on $M$. We denote with $\mathcal{M}$ the set of all the Borel probability measures on $M$, provided with the weak* topology, and with dist* a metric inducing this topology. The terms $\mathcal{M}_f$ and $\mathcal{E}_f$ denote the space of $f-$invariant Borel probability measures and the set of $f-$invariant ergodic Borel probability measures, respectively.

For each point $x \in M$, we define the empirical measures

$$\delta_{x,n} = \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)}$$

where $\delta_x$ is the Dirac measure at $x$. We denote with $\mathcal{V}_f(x)$ the set of all the Borel probabilities in $\mathcal{M}$ that are the weak* limits of the empirical measures. It is well known that $\mathcal{V}_f(x) \subset \mathcal{M}_f$.

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A sequence $\Phi = \{\phi_n\}_{n \geq 1}$ of continuous real functions is a subadditive potential on $M$, if

$$\phi_{n+m}(x) \leq \phi_n(x) + \phi_m(f^n x) \quad \text{for all } x \in M, \; n, m \in \mathbb{N}.$$

For $\mu \in \mathcal{M}_f$, it follows, from Kingman's sub-additive ergodic theorem (see [8] or [16, theorem 10.1]), that

$$\Phi^*(x) := \lim_{n \to \infty} \frac{1}{n} \phi_n(x) \; \text{ for } \mu - \text{a.e. } x \in M$$

and $\int \Phi^*(x) \mathrm{d}\mu = \inf_{n \geq 1} \frac{1}{n} \int \phi_n \mathrm{d}\mu$. The term $\Phi^*(x)$ is called *the growth rate* of the subadditive potentials $\Phi = \{\phi_n\}$ at $x$, defined as the existing limit for a set of full measure for any invariant measure. All along the paper we will assume that the growth rate is negative, i.e. $\Phi^*(x) < 0$ for $\mu-$almost all $x$ for one or more invariant measures $\mu$. We are interested to select other state points $x \in M$ for which the *largest growth rate* is still negative, namely:

$$\widetilde{\Phi}^*(x) := \limsup_{n \to \infty} \frac{1}{n} \phi_n(x) < 0.$$

The points $x \in M$ such that $\widetilde{\Phi}^*(x) < 0$, if exist, are called *optimal state points* for the given sequence $\Phi$ of subadditive potentials. We will say that the set of optimal state points is *observable,* if its Lebesgue measure is positive. We notice that we are neither assuming that the system preserves the Lebesgue measure, nor any measure that is absolutely continuous with respect to Lebesgue measure.

The problem of the abundance of optimal state points arises for example when studying the stability of linear control systems [4]. We are also interested to describe, and to find Lebesgue positive subsets of state points $x \in M$, that are not necessarily optimal, but for which the *smallest growth rate* is negative. Namely:

$$\widehat{\Phi}^*(x) := \liminf_{n \to \infty} \frac{1}{n} \phi_n(x) < 0.$$

Dai [5] gave a method to choose optimal points. We rewrite his result in our setting as the following theorem:

**Theorem 1.1.** ([5]) *Let $f : M \to M$ be a continuous map on a compact, finite-dimensional manifold $M$, $\mu$ an $f-$invariant ergodic measure, and $\Phi = \{\phi_n\}_{n \geq 1}$ a subadditive potential. If the growth rates of $\Phi$ satisfy*

$$\Phi^*(x) := \lim_{n \to \infty} \frac{1}{n} \phi_n(x) < 0 \quad \mu - a.e. \; x \in M$$

*then $\widetilde{\Phi}^*(x) < 0$ for all $x$ in the basin $B(\mu)$ of the measure $\mu$, where $B(\mu) := \{x \in M : \mathcal{V}_f(x) = \{\mu\}\}$.*

The above theorem states that all the points in the basin of an ergodic measure $\mu$ have negative largest growth rates. A natural question arises with respect to all the other invariant measures:

**Question 1:** *How can we ensure the existence of state points x that have negative largest growth rates, if the measure is f−invariant but not necessarily ergodic?*

For an experimenter it would be interesting to know whether the subadditive potentials have negative largest (or at least smallest) growth rates in a set with positive Lebesgue measure. To achieve that result and after Dai's Theorem it would be enough that the basin $B(\mu)$ of the ergodic measure $\mu$ has positive Lebesgue measure. But, although $B(\mu)$ has full $\mu$−measure, its Lebesgue measure may be zero, unless $\mu$ is *physical or SRB*. Nevertheless, general continuous dynamical systems may not have such a physical or SRB measure. Thus, it arises the following nontrivial question:

**Question 2:** *Under what conditions, even if no physical or SRB measure exists, the subadditive potentials have negative largest growth rates (or at least negative smallest growth rates) on a positive Lebesgue measure set?*

The following theorem provides a strong result in this direction. It was proved independently by Schreiber [10, theorem 1] and Sturman and Stark [13, theorem 1.7], and more recently Dai [5] gave a proof by a simple method. We rewrite it in our setting, with the following statement:

**Theorem 1.2.** ([10, 13]) *Let $f : M \to M$ be a continuous map on a compact, finite-dimensional manifold $M$, and $\Phi = \{\phi_n\}_{n \geq 1}$ a subadditive potential. If for each $f$−invariant ergodic measure $\mu$ the growth rates*

$$\Phi^*(x) := \lim_{n \to \infty} \frac{1}{n} \phi_n(x) < 0 \quad \mu - a.e. \ x \in M$$

*then $\widetilde{\Phi}^*(x) < 0$ for all $x \in M$.*

Cao [1] also extended the result in the latter theorem to random dynamical systems.

As said above, when observing the system the experimenter may just need to know that the subadditive potentials have negative growth rates in a positive Lebesgue measure set of state points $x$, instead in the whole manifold. That is why we are interested to find the conditions, weaker than the hypothesis of Theorem 1.2, to ensure that $\widetilde{\Phi}^*(x) < 0$, or at least $\widehat{\Phi}^*(x) < 0$, just for a Lebesgue-positive set of state points $x \in M$.

The questions 1 and 2 above are the motivations and tasks of this paper. To search for an answer to Question 2 we apply the recent works in [2] and [3]. Motivated and applying those results, and adapting the arguments of Dai [5], we give positive answers to Questions 1 and 2 in Theorems 2.9, 2.11 and 2.13, and Corollaries 3.1, 3.2 and 3.3 of this paper.

The exposition is organized as follows: Section 2 provides some preliminary definitions and the statements of our main results in Theorems 2.9, 2.11 and 2.13. Section 3 provides their detailed proofs and their corollaries. Section 4 gives additional remarks about the main results, and some examples to illustrate them.

## 2  Preliminaries and statements of the new results

This section provides some definitions and the statements of the main results of this paper.

To give a positive answer to question 1, we give the definition of strong basin of $\epsilon-$attraction of an invariant measure, as follows:

**Definition 2.1.** Given a probability measure $\mu \in \mathcal{M}$ and a real number $\epsilon > 0$, the following set

$$\mathcal{S}_\epsilon(\mu) := \{x \in M : \mathcal{V}_f(x) \subset \mathcal{N}_\epsilon(\mu)\}$$

is called *strong basin of $\epsilon-$attraction of $\mu$*, where $\mathcal{N}_\epsilon(\mu)$ denotes the $\epsilon-$neighborhood of $\mu$ under the metric dist$^*$. Furthermore, if the set $\mathcal{S}_\epsilon(\mu)$ has positive Lebesgue measure for all $\epsilon > 0$, then the measure $\mu$ is called *strong observable* for $f$.

To answer question 2, we first recall the definition of observable measure introduced in [2].

**Definition 2.2.** Given a probability measure $\mu \in \mathcal{M}$ and a real number $\epsilon > 0$, the following set

$$A_\epsilon(\mu) := \{x \in M : \mathcal{V}_f(x) \cap \mathcal{N}_\epsilon(\mu) \neq \emptyset\}$$

is called *basin of $\epsilon-$attraction of $\mu$*. Furthermore, if the set $A_\epsilon(\mu)$ has positive Lebesgue measure for all $\epsilon > 0$, then the measure $\mu$ is called *observable* for $f$. Let $\mathcal{O}_f$ denote the set of all the observable measures for $f$.

**Remark 2.3.** From the definitions above, it is immediate that any strong observable measure is observable. Nevertheless given an $f-$invariant measure $\mu$ its (strong) basin of $\epsilon-$attraction may be empty for all $\epsilon > 0$. If $\mu$ is ergodic then its strong basin of $0-$attraction (which is obviously included in its strong basin of $\epsilon-$attraction for all positive $\epsilon$) is not empty since it includes $\mu-$almost all the points.

It is easy to check that each (strong) observable measure is $f-$invariant, since it can be approximated by invariant measures and $\mathcal{M}_f$ is weak$^*$ compact. See [2] for more properties of observable measures. In order to give a class of systems for which the strong basins of $\epsilon-$attraction are not empty for all $\epsilon > 0$ and for all $\mu \in \mathcal{M}_f$, we first recall Bowen's specification property (see definition 18.3.8 in [7]):

**Definition 2.4.** A continuous map $f : M \to M$ satisfies *Bowen's specification property* if for each $\epsilon > 0$, there exists an integer $m = m(\epsilon)$ such that for any finite collection $\{I_j := [a_j, b_j] \subset \mathbb{N} : j = 1, 2, ..., k\}$ of finite intervals of natural numbers such that $a_{j+1} - b_j \geq m(\epsilon)$ for $j = 1, 2, ..., k-1$, for any $x_1, x_2, ..., x_k$ in $M$, and for any $p \geq b_k - a_1 + m(\epsilon)$, there exists a periodic point $x \in M$ of period at least $p$, satisfying

$$d(f^{l+a_j}(x), f^l(x_j)) < \epsilon \text{ for all } l = 0, 1, ..., b_j - a_j \text{ and every } j = 1, 2, ..., k.$$

**Proposition 2.5.** *Let $f : M \to M$ be a continuous map on a compact finite-dimensional manifold $M$. Assume that $f$ satisfies Bowen's specification property. Then for all $f-$invariant measure $\mu$ and for all $\epsilon > 0$,*

$$\mathcal{S}_\epsilon(\mu) \neq \emptyset \quad and \quad A_\epsilon(\mu) \neq \emptyset.$$

*Proof.* Since $\mathcal{S}_\epsilon(\mu) \subset A_\epsilon(\mu)$, it is enough to prove that $\mathcal{S}_\epsilon(\mu) \neq \emptyset$ for all $\epsilon > 0$. By hypothesis $f$ satisfies Bowen's specification property. Then, the set of ergodic measures $\mathcal{E}_f$ (precisely the subset of the invariant measures that are supported on periodic orbits) is dense in the space of $f-$invariant measures $\mathcal{M}_f$ (see the main theorem in [11] or [12] for a proof). Thus, for any $\mu \in \mathcal{M}_f$ and any $\epsilon > 0$, there exists an ergodic measure $\nu \in \mathcal{N}_\epsilon(\mu)$. The basin $B(\nu)$ is not empty since $\nu$ is ergodic and thus $\mathcal{V}_f(x) = \{\nu\}$ for $\nu$-a.e. point $x \in M$. Besides, from $\nu \in \mathcal{N}_\epsilon(\mu)$ we obtain that $B(\nu) \subset \mathcal{S}_\epsilon(\mu)$. We deduce that $\mathcal{S}_\epsilon(\mu) \neq \emptyset$, ending the proof. □

In order to answer question 2, we also need to revisit the definition of observable measures for a subsystem.

**Definition 2.6.** Let $B \subset M$ be a forward invariant set, i.e. $f(B) \subset B$, that has positive Lebesgue measure. A probability measure $\mu$ is *observable for $f \mid_B$*, if for all $\epsilon > 0$ the following set

$$A_\epsilon(B, \mu) := \{x \in B : \mathcal{V}_f(x) \cap \mathcal{N}_\epsilon(\mu) \neq \emptyset\}$$

has positive Lebesgue measure. Let $\mathcal{O}_{f\mid_B}$ denote the set of all the observable measures for $f \mid_B$.

The following definition of Milnor-like attractor was introduced in [3].

**Definition 2.7.** Let $K \subset M$ be a nonempty, compact and $f-$invariant set, i.e. $f^{-1}(K) = K$. We say that $K$ is a *Milnor-like attractor* if the following set

$$B(K) := \{x \in M : \liminf_{n \to \infty} \frac{1}{n}\sharp\{0 \le j \le n-1 : f^j(x) \in \mathcal{N}_\epsilon(K)\} = 1, \forall \epsilon > 0\}$$

has positive Lebesgue measure, where $\sharp A$ denotes the cardinality of a set $A$ and $\mathcal{N}_\epsilon(K)$ denotes the $\epsilon-$neighborhood of $K$, i.e., $\mathcal{N}_\epsilon(K) := \bigcup_{x \in K} B(x, \epsilon)$ and $B(x, \epsilon)$ is a ball of radius $\epsilon$ centered at $x$. The set $B(K)$ is called the basin of $K$.

Note that the lim inf equal to one in the definition above, implies that the limit exists and is equal to one.

For each $0 < \alpha \le 1$, the Milnor-like attractor $K$ is called $\alpha-observable$ if $m(B(K)) \ge \alpha$. An $\alpha-$observable Milnor-like attractor *is minimal,* if it has no proper subsets that are also $\alpha-$observable Milnor-like attractor for the same value of $\alpha$.

We restate here, just for completeness, the following result (see [2, 3] for a proof):

**Theorem 2.8.** ( [2, 3])*Let $f : M \to M$ be a continuous map on a compact, finite-dimensional manifold $M$, and let $B \subset M$ be a forward invariant set that has positive Lebesgue measure. Then the following properties hold:*

(1) *The set $\mathcal{O}_f$ of all observable measures for $f$ is nonempty, and minimally weak\* compact containing for Lebesgue almost all $x \in M$, all the weak\* limits of the convergent subsequences of empirical measures.*

(2) *For each $0 < \alpha \leq 1$, there exists a minimal $\alpha-$observable Milnor-like attractor.*

(3) *The set $\mathcal{O}_{f|_B}$ is weak\* compact and nonempty.*

(4) *The set $\mathcal{O}_{f|_B}$ is the minimal weak\* compact set in the space $\mathcal{M}$ such that $\mathcal{V}_f(x) \subset \mathcal{O}_{f|_B}$ for Lebesgue almost all $x \in B$.*

Now we state the main theorems of this paper. We will give their proofs in the next section.

The first theorem states that a subadditive potential has negative largest growth rates at all the state points $x$ belonging to the strong basin of $\epsilon-$attraction of an invariant measure, for some $\epsilon > 0$ (and thus for all $\epsilon > 0$ small enough).

**Theorem 2.9.** *Let $f : M \to M$ be a continuous map on a compact, finite-dimensional manifold $M$, $\mu$ an $f-$invariant measure, and $\Phi = \{\phi_n\}_{n \geq 1}$ a subadditive potential. If the growth rates of $\Phi$ satisfy*

$$\Phi^*(x) := \lim_{n \to \infty} \frac{1}{n} \phi_n(x) < 0 \quad \mu - a.e. \ x \in M,$$

*then there exists $\epsilon > 0$ such that $\widetilde{\Phi}^*(x) < 0$ for each point $x$ in $\mathcal{S}_\epsilon(\mu)$.*

**Remark 2.10.** Combined with Proposition 2.5 (which asserts that $\mathcal{S}_\epsilon(\mu) \neq \emptyset$) the theorem above answers positively Question 1 of the introduction, for any system that satisfies Bowen's specification property and for any invariant measure $\mu$.

The second theorem states that the subadditive potentials have negative *smallest growth rates* in the basin of $\epsilon-$attraction of any invariant measure, for some $\epsilon > 0$.

**Theorem 2.11.** *Let $f : M \to M$ be a continuous map on a compact, finite-dimensional manifold $M$, $\mu$ an $f-$invariant measure, and $\Phi = \{\phi_n\}_{n \geq 1}$ a subadditive potential. If the growth rates of $\Phi$ satisfies*

$$\Phi^*(x) := \lim_{n \to \infty} \frac{1}{n} \phi_n(x) < 0 \quad \mu - a.e. \ x \in M$$

*then there exists $\epsilon > 0$ such that $\widehat{\Phi}^*(x) < 0$ for each point $x$ in $A_\epsilon(\mu)$.*

**Remark 2.12.** From theorem 2.8 we deduce that there always exist observable invariant measures $\mu$. Applying theorem 2.11 to those measures, we obtain that the *smallest growth rates* of the subadditive potential is negative on a set of positive Lebesgue measure, for any continuous system.

Moreover, the following theorem states that under mild stronger conditions the *largest growth rates* are also negative for Lebesgue almost all the points in the basin of any Milnor-like attractor:

**Theorem 2.13.** *Let $f : M \to M$ be a continuous map on a compact, finite-dimensional manifold $M$, and $\Phi = \{\phi_n\}_{n \geq 1}$ a subadditive potential. Assume that $K$ is an $\alpha-$observable Milnor-like attractor for some $0 < \alpha \leq 1$, and $B(K)$ is its basin. If to each observable measure $\mu \in \mathcal{O}_{f|_{B(K)}}$, the growth rates*

$$\Phi^*(x) := \lim_{n \to \infty} \frac{1}{n} \phi_n(x) < 0 \quad \mu - a.e.\ x \in M$$

*then $\widetilde{\Phi}^*(x) < 0$ for $m-$almost every $x \in B(K)$.*

**Remark 2.14.** Note that the Lebesgue measure of the basin $B(K)$ of the Milnor-like attractor $K$ in the latter theorem is larger than or equal to $\alpha$. Thus the *largest growth rates* of the subadditive potentials is negative on a set with Lebesgue measure that is at least equal to $\alpha$. This implies that the optimal state points $x$ (namely the points for which the largest growth rates are negative) cover a set that is Lebesgue $\alpha$-observable in the manifold $M$. This answers positively Question 2 of the Introduction.

Moreover, if $K$ is a $1-$observable Milnor-like attractor (such $K$ always exists after part (2) of theorem 2.8), then Theorem 2.13 asserts that the *largest growth rates* of the subadditive potentials are negative Lebesgue almost everywhere.

To end this section, we state a useful known lemma. It appears in many places; see for example [1]. We give a proof here just for completeness.

**Lemma 2.15.** *Let $f : M \to M$ be a continuous map on a compact, finite-dimensional manifold $M$, and $\Phi = \{\phi_n\}_{n \geq 1}$ a subadditive potential. Fix any positive integer $l$. Then*

$$\phi_n(x) \leq C + \sum_{i=0}^{n-1} \frac{1}{l} \phi_l(f^i x) \quad \forall x \in M$$

*where $C$ is a constant depending only on $l$.*

*Proof.* Fix a positive integer $l$. For each natural number $n$, we write $n = sl + k$, where $0 \leq s, 0 \leq k < l$. Then, for any integer $0 \leq j < l$ we have

$$\phi_n(x) \leq \phi_j(x) + \phi_l(f^j x) + \cdots + \phi_l(f^{(s-2)l} f^j x) + \phi_{k+l-j}(f^{(s-1)l} f^j x),$$

where $\phi_0(x) \equiv 0$. Let $C_1 = \max_{j=1,\cdots 2l} ||\phi_j||_\infty$. Adding $\phi_n(x)$ when $j$ takes all the natural values from 0 to $l-1$, we have

$$l\phi_n(x) \leq 2lC_1 + \sum_{i=0}^{(s-1)l-1} \phi_l(f^i x).$$

Hence

$$\phi_n(x) \leq 2C_1 + \sum_{i=0}^{(s-1)l-1} \frac{1}{l}\phi_l(f^i x) \leq 4C_1 + \sum_{i=0}^{n-1} \frac{1}{l}\phi_l(f^i x).$$

Choosing $C = 4C_1$ the desired result follows. □

# 3 Proofs of the main results

This section provides the proofs of the theorems in section 2.

## 3.1 Proof of theorem 2.9

*Proof.* Let $\mu$ be an $f-$invariant measure that satisfies the hypothesis of theorem 2.9. The arguments here are similar to those of Dai in [5]. Let $\psi_n(x) = \max\{-n, \phi_n(x)\}$ for all $n \geq 1$ and each $x \in M$. It is easy to see that the sequence of functions $\Psi = \{\psi_n\}$ is subadditive. Set

$$\widetilde{\Psi}^*(x) := \limsup_{n\to\infty} \frac{1}{n}\psi_n(x) \quad \forall x \in M.$$

Under the hypothesis of theorem 2.9, it follows that $\widetilde{\Psi}^*(x) < 0$ for $\mu-$almost every $x \in M$. Since $\psi_n(x) \geq \phi_n(x)$ for all $n \geq 1$ and all $x \in M$, to prove theorem 2.9 it is enough to show that $\widetilde{\Psi}^*(x) < 0$ for all $x \in \mathcal{S}_\epsilon(\mu)$ for some $\epsilon > 0$.

Using the definition of $\Psi$ and the subadditivity of $\Psi$, we have

$$-1 \leq \frac{1}{n}\psi_n(x) \leq ||\psi_1||_\infty \quad \forall x \in M.$$

It follows from the Fatou lemma that

$$\inf_{n\geq 1} \frac{1}{n}\int \psi_n \mathrm{d}\mu = \lim_{n\to\infty} \frac{1}{n}\int \psi_n \mathrm{d}\mu \leq \int \limsup_{n\to\infty} \frac{1}{n}\psi_n(x)\mathrm{d}\mu = \int \widetilde{\Psi}^*(x)\mathrm{d}\mu < 0.$$

Therefore, there exists an integer $l \geq 1$ such that

$$-1 \leq \frac{1}{l}\int \psi_l \mathrm{d}\mu < 0.$$

For some sufficiently small $\eta > 0$, say $\eta < \frac{|\frac{1}{l}\int \psi_l \mathrm{d}\mu|}{2}$, fix a positive number $\epsilon > 0$ such that

$$\mathrm{dist}^*(\mu, \nu) \leq \epsilon \Rightarrow |\int \frac{1}{l}\psi_l \mathrm{d}\mu - \int \frac{1}{l}\psi_l \mathrm{d}\nu| < \eta.$$

If the strong basin of $\epsilon$-attraction of $\mu$ is empty, i.e., $\mathcal{S}_\epsilon(\mu) = \emptyset$, then there is nothing to prove. Otherwise, let

$$\mathcal{D} = \{x \in \mathcal{S}_\epsilon(\mu) : \limsup_{n \to \infty} \frac{1}{n} \psi_n(x) \geq 0\}.$$

We will prove that $\mathcal{D} = \emptyset$. Assume by contradiction that there exists $x_0 \in \mathcal{D}$. Since $x_0 \in \mathcal{D} \subset \mathcal{S}_\epsilon(\mu)$, choose a subsequence of integers $\{n_i\}$ such that $\delta_{x_0, n_i}$ converges weakly to a measure $\tilde{\mu}$ and $\lim_{i \to \infty} \frac{1}{n_i} \psi_{n_i}(x_0) = \limsup_{n \to \infty} \frac{1}{n} \psi_n(x_0)$. Note that $\tilde{\mu} \in \mathcal{V}_f(x_0) \subset \mathcal{N}_\epsilon(\mu)$, i.e., $\text{dist}^*(\mu, \tilde{\mu}) \leq \epsilon$. . It follows that

$$0 > \int \frac{1}{l} \psi_l \mathrm{d}\mu + \eta > \int \frac{1}{l} \psi_l \mathrm{d}\tilde{\mu} = \lim_{i \to \infty} \int \frac{1}{l} \psi_l \mathrm{d}\delta_{x_0, n_i} = \lim_{i \to \infty} \frac{1}{n_i} \sum_{j=0}^{n_i - 1} \frac{1}{l} \psi_l(f^j x_0).$$

Using lemma 2.15, we have

$$\lim_{i \to \infty} \frac{1}{n_i} \psi_{n_i}(x_0) \leq \lim_{i \to \infty} \frac{1}{n_i} \sum_{j=0}^{n_i - 1} \frac{1}{l} \psi_l(f^j x_0).$$

Note that $x_0 \in \mathcal{D}$, we have

$$0 > \lim_{i \to \infty} \frac{1}{n_i} \sum_{j=0}^{n_i - 1} \frac{1}{l} \psi_l(f^j x_0) \geq \lim_{i \to \infty} \frac{1}{n_i} \psi_{n_i}(x_0) \geq 0$$

which is a contradiction. This completes the proof of theorem 2.9. $\qquad\square$

**Corollary 3.1.** *Let $f : M \to M$ be a continuous map on a compact, finite-dimensional manifold $M$, and $\Phi = \{\phi_n\}_{n \geq 1}$ a subadditive potential. Assume that there exists a strong observable measure $\mu$. If the growth rates of $\Phi$ satisfies*

$$\Phi^*(x) := \lim_{n \to \infty} \frac{1}{n} \phi_n(x) < 0 \quad \mu - a.e. \ x \in M$$

*then $\widetilde{\Phi}^*(x) < 0$ on a set with positive Lebesgue measure.*

*Proof.* First note that $\mu$ is also $f$−invariant, thus there exists $\epsilon > 0$ such that $\widetilde{\Phi}^*(x) < 0$ on the set $\mathcal{S}_\epsilon(\mu)$, i.e., the strong basin of $\epsilon$−attraction of $\mu$. And since $\mu$ is strong observable, we have $m(\mathcal{S}_\epsilon(\mu)) > 0$. This completes the proof of the corollary. $\qquad\square$

## 3.2 Proof of theorem 2.11

*Proof.* Let $\mu$ be an $f$−invariant measure that satisfies the hypothesis of theorem 2.11. Let $\psi_n(x) = \max\{-n, \phi_n(x)\}$ for all $n \geq 1$ and each $x \in M$. As in the previous proof, the sequence $\Psi = \{\psi_n\}$ is subadditive. Set

$$\widehat{\Psi}^*(x) := \liminf_{n \to \infty} \frac{1}{n} \psi_n(x) \quad \forall x \in M.$$

Under the hypothesis of theorem 2.11, it follows that $\widehat{\Psi}^*(x) < 0$ for $\mu-$almost all $x$. Since $\psi_n(x) \geq \phi_n(x)$ for all $n \geq 1$ and all $x \in M$, to prove theorem 2.11 it is enough to show that $\widehat{\Psi}^*(x) < 0$ for each point in $A_\epsilon(\mu)$ for some $\epsilon > 0$.

Using the definition of $\Psi$ and the subadditivity of $\Psi$, we have

$$-1 \leq \frac{1}{n}\psi_n(x) \leq ||\psi_1||_\infty \quad \forall x \in M.$$

It follows from the Fatou lemma that

$$\inf_{n \geq 1} \frac{1}{n} \int \psi_n \mathrm{d}\mu = \lim_{n \to \infty} \frac{1}{n} \int \psi_n \mathrm{d}\mu \leq \int \limsup_{n \to \infty} \frac{1}{n}\psi_n(x)\mathrm{d}\mu < 0.$$

The last inequality holds since $\limsup\limits_{n \to \infty} \frac{1}{n}\psi_n(x) < 0$ for $\mu-$almost all $x \in M$. Therefore, there exists an integer $l \geq 1$ such that

$$-1 \leq \frac{1}{l} \int \psi_l \mathrm{d}\mu < 0.$$

For some sufficiently small $\eta > 0$, say $\eta < \frac{|\frac{1}{l}\int \psi_l \mathrm{d}\mu|}{2}$, fix a positive number $\epsilon > 0$ such that

$$\mathrm{dist}^*(\mu, \nu) \leq \epsilon \Rightarrow |\int \frac{1}{l}\psi_l \mathrm{d}\mu - \int \frac{1}{l}\psi_l \mathrm{d}\nu| < \eta.$$

If the basin of $\epsilon$-attraction of $\mu$ is empty, i.e., $A_\epsilon(\mu) = \emptyset$, then there is nothing to prove. Otherwise, let

$$\mathcal{D} = \{x \in A_\epsilon(\mu) : \liminf_{n \to \infty} \frac{1}{n}\psi_n(x) \geq 0\}.$$

We will prove that $\mathcal{D} = \emptyset$. Assume by contradiction that there exists $x_0 \in \mathcal{D}$. Since $x_0 \in \mathcal{D} \subset A_\epsilon(\mu)$, there exists $\tilde{\mu} \in \mathcal{V}_f(x_0)$ such that $\mathrm{dist}^*(\mu, \tilde{\mu}) \leq \epsilon$. Choose a subsequence of integers $\{n_i\}$ such that $\delta_{x_0,n_i}$ converges weakly to the measure $\tilde{\mu}$. It follows that

$$0 > \int \frac{1}{l}\psi_l \mathrm{d}\mu + \eta > \int \frac{1}{l}\psi_l \mathrm{d}\tilde{\mu} = \lim_{i \to \infty} \int \frac{1}{l}\psi_l \mathrm{d}\delta_{x_0,n_i} = \lim_{i \to \infty} \frac{1}{n_i}\sum_{j=0}^{n_i-1}\frac{1}{l}\psi_l(f^j x_0).$$

Using lemma 2.15, we have

$$\liminf_{i \to \infty} \frac{1}{n_i}\psi_{n_i}(x_0) \leq \lim_{i \to \infty} \frac{1}{n_i}\sum_{j=0}^{n_i-1}\frac{1}{l}\psi_l(f^j x_0).$$

Note that $x_0 \in \mathcal{D}$, we have

$$0 > \lim_{i \to \infty} \frac{1}{n_i}\sum_{j=0}^{n_i-1}\frac{1}{l}\psi_l(f^j x_0) \geq \liminf_{i \to \infty} \frac{1}{n_i}\psi_{n_i}(x_0) \geq \liminf_{n \to \infty} \frac{1}{n}\psi_n(x_0) \geq 0$$

which is a contradiction. This completes the proof of theorem 2.11. $\qquad\square$

**Corollary 3.2.** *Let $f : M \to M$ be a continuous map on a compact, finite-dimensional manifold $M$, and $\Phi = \{\phi_n\}_{n \geq 1}$ a subadditive potential. Let $\mu$ be any (always existing) observable measure for $f$. If the growth rates of $\Phi$ satisfy*

$$\Phi^*(x) := \lim_{n \to \infty} \frac{1}{n} \phi_n(x) < 0 \quad \mu - a.e.\ x \in M$$

*then $\widehat{\Phi}^*(x) < 0$ on a set with positive Lebesgue measure.*

*Proof.* First note that $\mu$ is also $f-$invariant, thus there exists $\epsilon > 0$ such that $\widehat{\Phi}^*(x) < 0$ on the set $A_\epsilon(\mu)$, i.e., the basin of $\epsilon-$attraction of $\mu$. And since $\mu$ is observable, we have $m(A_\epsilon(\mu)) > 0$. This completes the proof of the corollary. $\qquad \square$

## 3.3 Proof of theorem 2.13

*Proof.* As in the proof of theorem 2.11, define $\psi_n(x) = \max\{-n, \phi_n(x)\}$ for all $n \geq 1$ and each $x \in M$. Then the sequence of functions $\Psi = \{\psi_n\}$ is a family of continuous functions which is subadditive. Set

$$\widetilde{\Psi}^*(x) := \limsup_{n \to \infty} \frac{1}{n} \psi_n(x) \quad \forall x \in M.$$

Under the hypothesis of theorem 2.13, to each observable measure $\mu \in \mathcal{O}_{f|_B}$, it holds that $\widetilde{\Psi}^*(x) < 0$ for $\mu-$almost every $x \in M$. By the definition of $\Psi$, it is easy to see that $\widetilde{\Psi}^*(x) \geq \widehat{\Phi}^*(x)$ for each $x \in M$. So, to prove theorem 2.13 it is enough to show that $\widetilde{\Psi}^*(x) < 0$ for $m-$almost every $x \in B(K)$, where $m$ denotes the Lebesgue measure.

Since the Milnor-attractor $K$ is $\alpha-$observable, its basin $B(K)$ satisfies $m(B(K)) \geq \alpha$. It is easy to check that the basin $B(K)$ is forward invariant. Thus, each of the observable measures in the set $\mathcal{O}\mid_{f|_{B(K)}}$ satisfies Theorem 2.8.

Let $\mathcal{D} = \{x \in B(K) : \limsup_{n \to \infty} \frac{1}{n} \psi_n(x) \geq 0\}$. To end the proof of Theorem 2.13 it is now enough to show that $m(\mathcal{D}) = 0$.

Assume by contradiction that $m(\mathcal{D}) > 0$. By the fourth item of theorem 2.8, we can choose a point $x_0 \in \mathcal{D}$ such that $\mathcal{V}_f(x_0) \subset \mathcal{O}_{f|_{B(K)}}$. We can take a subsequence of integers $\{n_i\}$ such that $\delta_{x_0,n_i}$ converges weakly to the measure $\mu \in \mathcal{V}_f(x_0)$ and $\lim_{i \to \infty} \frac{1}{n_i} \psi_{n_i}(x_0) = \limsup_{n \to \infty} \frac{1}{n} \psi_n(x_0)$. As

$$-1 \leq \frac{1}{n} \psi_n(x) \leq ||\psi_1||_\infty \quad \forall x \in M,$$

it follows from the Fatou lemma that

$$\inf_{n \geq 1} \frac{1}{n} \int \psi_n \mathrm{d}\mu = \lim_{n \to \infty} \frac{1}{n} \int \psi_n \mathrm{d}\mu \leq \int \limsup_{n \to \infty} \frac{1}{n} \psi_n(x) \mathrm{d}\mu < 0$$

since $\mu \in \mathcal{O}_{f|_{B(K)}}$, and by the definition of $\Psi$ and the hypothesis of theorem 2.13, we obtain that

$$\limsup_{n \to \infty} \frac{1}{n} \psi_n(x) < 0$$

for $\mu-$almost every point $x \in M$. So, there exists an integer $l \geq 1$ such that $-1 \leq \frac{1}{l} \int \psi_l \mathrm{d}\mu < 0$, and further

$$0 > \frac{1}{l} \int \psi_l \mathrm{d}\mu = \lim_{i \to \infty} \int \frac{1}{l} \psi_l \mathrm{d}\delta_{x_0, n_i} = \lim_{i \to \infty} \frac{1}{n_i} \sum_{j=0}^{n_i - 1} \frac{1}{l} \psi_l(f^j x_0).$$

Using lemma 2.15, we have

$$\lim_{i \to \infty} \frac{1}{n_i} \psi_{n_i}(x_0) \leq \lim_{i \to \infty} \frac{1}{n_i} \sum_{j=0}^{n_i - 1} \frac{1}{l} \psi_l(f^j x_0).$$

Note that $x_0 \in \mathcal{D}$, we have

$$0 > \lim_{i \to \infty} \frac{1}{n_i} \sum_{j=0}^{n_i - 1} \frac{1}{l} \psi_l(f^j x_0) \geq \lim_{i \to \infty} \frac{1}{n_i} \psi_{n_i}(x_0) \geq 0$$

which is a contradiction. This completes the proof of theorem 2.13. $\qquad \square$

Using the first item of theorem 2.8 and the same arguments as in the proof of theorem 2.13, we have the following corollary.

**Corollary 3.3.** *Let $f : M \to M$ be a continuous map on a compact, finite-dimensional manifold $M$, and $\Phi = \{\phi_n\}_{n \geq 1}$ a subadditive potential. If for all observable measure $\mu \in \mathcal{O}_f$, the growth rates*

$$\Phi^*(x) := \lim_{n \to \infty} \frac{1}{n} \phi_n(x) < 0 \quad \mu - a.e. \ x \in M$$

*then $\widetilde{\Phi}^*(x) < 0$ for Lebesgue almost all $x \in M$.*

## 4  Examples and additional remarks

In [2] it is proved that observable measures exist for all continuous systems. Nevertheless, the following example (attributed to Bowen [6, 15] and early cited in [14]) shows that not all continuous dynamical systems (indeed not all $C^2$ systems) have strong observable measures. So, in this example Corollary 3.1 can not be applied. Nevertheless we will prove that it still satisfies the final assertion of that Corollary, since for Lebesgue almost all $x \in M$, the growth rate $\widetilde{\Phi}^*(x) < 0$.

**Example 4.1.** Consider a $C^2$ diffeomorphism $f$ in a compact ball $M$ of $\mathbb{R}^2$ with two hyperbolic saddle points $A$ and $B$ in the boundary $\partial M$ of $M$ such that (half) the unstable global manifold $W^u(A) \setminus \{A\}$ is an embedded $C^2$ arc that coincides with (half) the stable global manifold $W^s(B) \setminus \{B\}$, conversely $W^s(A) \setminus \{A\} = W^u(B) \setminus \{B\}$, and besides $\partial M = W^u(A) \cup W^u(B)$. Take $f$ such that there exists a source $C \in U$ where $U$ is the topological open ball with boundary $W^u(A) \cup W^u(B)$. One can choose $f$ such that for all

$x \in U$ the $\alpha-$limit is $\{C\}$ and the $\omega-$limit contains $\{A, B\}$. See figure 1 in [15]. If the eigenvalues of the derivative of $f$ at $A$ and $B$ are adequately chosen as specified in [6, 15], then the sequence of empirical measures for any $x \in U \setminus \{C\}$ is not convergent. It has at least two subsequences convergent to different convex combinations of the Dirac measures $\delta_A$ and $\delta_B$. The systems, as proved in [6], satisfies the following property:

There exists a segment $\Gamma$ in the space of $f-$invariant measures, such that $\Gamma$ is a family of convex combinations of the two Dirac measures $\delta_A$ and $\delta_B$, and $\mathcal{V}_f(x) = \Gamma$ for Lebesgue almost all points $x$.

Therefore, as a corollary of the result above, we obtain:

**Proposition 4.2.** *In the example* 4.1:

(A) *For Lebesgue almost all points, the sequence of empirical measures does not converge.*

(B) *All measures in $\Gamma$ are observable according to definition 2.2 and are the only observable measures,*

(C) *There does not exist strong observable measures according to definition 2.1.*

*Proof.* (A) is immediate from the fact that $\mathcal{V}_f(x) = \Gamma$ for Lebesgue almost all points $x$. We refer the proof of (B) to the example 5.5 in [2]. Finally, let us prove (C). Since for all $x \in U \setminus \{C\}$ the sequence of empirical measures has at least two subsequences convergent to different convex combinations of the Dirac measures $\delta_A$ and $\delta_B$, no invariant measure satisfies Definition 2.1 of strong observability. In other words, there does not exist strong observable measures, because for any invariant measure the strong basin of $\epsilon-$attraction is empty. $\qquad\square$

**Remark 4.3.** The proposition above shows that there exist dynamical systems for which Theorem 2.9 and Corollary 3.1 do not give information about the existence of optimal state points for the subadditive potentials. Nevertheless, if $\Phi^*(x) < 0$ for $\mu$-a.e. just for one (not necessarily ergodic) invariant measure $\mu$, and if $\mu$ is some of the always existing observable measures, then for those systems Theorem 2.11 and Corollary 3.2 still ensure the existence of a Lebesgue positive set of state points with negative smallest growth rates.

Moreover, in Example 4.1 we still have the following very strong statement:

**Proposition 4.4.** *Let $f$ be the Bowen's example defined in* Example 4.1, *$\mu$ an observable measure for $f$, and $\Phi$ a subadditive potential with the growth rate $\Phi^*(x) < 0$ for $\mu$-a.e. $x$. Then, the largest growth rate $\widetilde{\Phi}^*(x)$ is negative for Lebesgue almost all $x \in M$.*

*Proof.* Any observable measure in this example, i.e. any $\mu \in \Gamma$, has exactly two ergodic components, that are $\delta_A$ and $\delta_B$, and has basin of $\epsilon-$attraction $A_\epsilon(\mu)$ that covers Lebesgue almost all $M$. Since $\Phi^*(x) < 0$ for $\mu$-a.e. $x$, $\Phi^*(A) < 0$ and $\Phi^*(B) < 0$, because $\mu$ is supported on $\{A, B\}$. Therefore $\Phi^*(x) < 0$ $\nu$-a.e. for all other observable measure $\nu$, because $\nu$ is a convex combination of $\delta_A$ and $\delta_B$. After Corollary 3.3, the largest growth rate $\widetilde{\Phi}^*(x)$ is negative for Lebesgue almost all $x \in M$. $\qquad\square$

**Remark 4.5.** The proof above shows how Theorem 2.13 and its Corollary 3.3 are powerful results, particularly useful if neither physical nor strong observable measure exists. In fact, even if the set of the observable measures components is uncountable (as in Example 4.1), the set of all their ergodic components may be finite and still the conclusion that Lebesgue almost all points are optimal states for a given subadditive potential, may hold. We recall from Definition 2.1 that if no strong observable measure exists, then no physical measure exists. And if no physical measure exists, then the (never empty) set of observable measures is necessarily uncountable (for a proof see [2]).

**Example 4.6.** In theorem 3.4 of [9] Misiurewicz proved that there exists a $C^0$ topologically expanding map $f$ in the circle $S^1$ such that for Lebesgue almost all $x \in S^1$ the limit set $\mathcal{V}_f(x)$ of the sequence of empirical measures is composed by all the (uncountably infinitely many) $f-$invariant measures. Thus, $\mathcal{V}_f(x) = \mathcal{O}_f = \mathcal{M}_f$ for Lebesgue almost all $x \in S^1$. Then, it is easy to check that there is no strong observable measure in this example.

**Proposition 4.7.** *For the example* 4.6 *of Misiurewicz, there exist two observable ergodic invariant measures $\mu$ and $\nu$, and a subadditive potential $\Phi = \{\phi_n\}_n$, such that the following properties hold:*

(i) $\Phi^*(x) := \lim_{n\to\infty} \frac{1}{n}\phi_n(x) < 0$ *for $\mu-$a.e. $x$.*

(ii) *The smallest growth rates $\widehat{\Phi}^*(x) < 0$ for all $x \in A_\epsilon(\mu)$ for all $\epsilon > 0$ small enough.*

(iii) *The largest growth rates $\widetilde{\Phi}^*(x) > 0$ for all $x \in A_\epsilon(\nu)$ for all $\epsilon > 0$ small enough.*

*Proof.* The map $f$ of Misiurewicz has a dense set of periodic orbits (see Theorem 3.4 of [9]). Choose two of those periodic orbits, say $\mathcal{O}_1$ and $\mathcal{O}_2$ and a real continuous function $g : S^1 \to \mathbb{R}$ such that $g(x) = -1$ for all $x \in \mathcal{O}_1$ and $g(x) = 1$ for all $x \in \mathcal{O}_2$.

Define $\phi_n(x) = \sum_{i=0}^{n-1} g(f^i x)$. Note that $\Phi = \{\phi_n\}_{n\geq 1}$ is an additive potential, since $\phi_{n+m} = \phi_n + \phi_m(f^n)$ for all natural numbers $n$ and $m$. Therefore $\{\phi_n\}_n$ and $\{-\phi_n\}_n$ are also subadditive potentials.

The measure $\mu$ supported on $\mathcal{O}_1$ and equally distributed in all the points of $\mathcal{O}_1$ is ergodic. And besides it is observable because all invariant measures are observable for $f$. Furthermore, by construction $\frac{1}{n}\phi_n(x) = -1$ for all $x \in \mathcal{O}_1$, so $\Phi^*(x) = -1 < 0$ for $\mu-$a.e. $x$, proving (i). Therefore, applying theorem 2.11, the assertion (ii) follows for some $\epsilon > 0$, and after the definition of basin $A_\epsilon(\mu)$ of $\epsilon$- attraction, the assertion (ii) is proved for all $\epsilon > 0$ small enough.

Now it is left to prove (iii). Consider the measure $\nu$ supported on $\mathcal{O}_2$ and equally distributed in all the points of $\mathcal{O}_2$. Similar arguments to those used with $\mu$ lead to the conclusion that $\nu$ is observable ergodic and $\Phi^*(x) = +1$ for all $x \in \mathcal{O}_2$. Besides, for all $x \in \mathcal{O}_2$, $-\Phi^*(x) = \Psi^*(x)$, where $\Psi := \{-\phi_n\}_{n\geq 1}$. Therefore, $\Psi^*(x) = -1$ for $\nu$-a.e. point. Applying again theorem 2.11 we obtain that $\widehat{\Psi}^*(x) < 0$ for all $x \in A_\epsilon(\nu)$, for all $\epsilon > 0$ small enough. Since $\widehat{\Psi}^*(x) = -\widetilde{\Phi}^*(x)$, we conclude that $\widetilde{\Phi}^*(x) > 0$ for all $x \in A_\epsilon(\nu)$ for all $\epsilon > 0$ small enough. This completes the proof of (iii). $\qquad\square$

**Corollary 4.8.** *In the example* 4.6 *of Misiurewicz, if* $\Phi$ *is the subadditive potentials of Proposition* 4.7, *then* $\Phi^*(x) < 0$ *for* $\mu$-*a.e., but for Lebesgue almost all points* $x \in S^1$:

$$\widehat{\Phi}^*(x) < 0, \quad \widetilde{\Phi}^*(x) > 0.$$

*Proof.* Since $\mathcal{V}_f(x) = \mathcal{M}_f$ for Lebesgue almost all point $x \in S^1$, $\mathcal{V}_f(x)$ intersects $A_\epsilon(\mu)$ for all $\epsilon > 0$ for Lebesgue almost all $x$ and for all $\mu \in \mathcal{M}_f$. This implies that for all positive $\epsilon$ and for all pair of invariant measures $\mu$ and $\nu$, $A_\epsilon(\mu) = A_\epsilon(\nu) = S^1$ up to sets of zero Lebesgue measures. Hence, the assertion (ii) of Proposition 4.7 implies that $\widehat{\Phi}^*(x) < 0$ for Lebesgue almost all $x \in S_1$. Analogously the assertion (iii) implies that $\widetilde{\Phi}^*(x) > 0$ for Lebesgue almost all $x \in S_1$. $\qquad\qquad\square$

**Remark 4.9.** After the Corollary above, the example 4.6 of Misiurewicz shows that the assertion $\Phi^*(x) < 0$ for $\mu$ a.e., assumed in the hypothesis of Theorem 2.11 and Corollary 3.2 can be satisfied, but the conclusions of those two results stating that the smallest growth rate is negative, can not be improved, since the largest growth rates may be positive, as in this concrete example, for Lebesgue almost all $x$ in the manifold.

Therefore, the last example shows that the conclusion of Theorem 2.11 can not be strengthened in general. In this sense Theorem 2.11 and Corollary 3.2 are optimally stated if one wishes them to hold for all the continuous systems.

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