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Ergodic Theorems with Respect to Lebesgue

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Preliminary version of the abstract, introduction and references.
The final complete version is published by
World Science and Engineering Academy and Society
in the peer reviewed journal
WSEAS Transactions on Mathematics (ISSN 11092769), Volume 10, Issue 12 , year 2011, pages 463–479,
http://www.wseas.us/e-library/transactions/mathematics/2011/53-746.pdf
(free access to the full-text pdf-file in the site of the publisher)

Abstract

We study, from the ergodic viewpoint, the asymptotic dynamics in the future of a full Lebesgue set of initial states. The dynamical systems under research are deterministic and evolving with discrete time $n \in \mathbb{N}$ by the forward iterations of any continuous map $f : M \mapsto M$ acting on a finite-dimensional, compact and Riemannian manifold M. First, we revisit the classic definition of physical or SRB probability measures, and its generalized notion of weak physical probabilities. Then, inspired in the statistical meaning of the ergodic attractors defined by Pugh and Schub, which support ergodic physical measures, we define the more general concept of ergodic-like attractor. We prove that any such generalized attractor is the support of weak physical probabilities and conversely. Then, we revisit the proof of existence of weak physical probabilities and conclude that any continuous dynamics exhibits at least one ergodic-like attractor.

Key-Words: Ergodic theory, physical measures, ergodic attractors, topological dynamics, theoretical measure dynamics

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1 Introduction

We consider, as the phase space where a dynamical system evolves, any finite dimensional, compact and Riemannian manifold M. We investigate the dynamics, evolving on M deterministically in the future, with discrete time $n \in \mathbb{N}$. Precisely, the system is obtained by iteration $f^n := f \circ f \circ \ldots \circ f$, for all $n \geq 0$, of a continuous map $f : M \mapsto M$. In the sequel we refer to it, as *a continuous dynamical system,* and denote it in brief, with f. We will focus in the abstract general scenario of all the continuous dynamical systems, from the viewpoint of the Ergodic Theory. Namely, we will search for theoretical probability measures, determined by the dynamics in regime, i.e. f-invariant, that are representative of the asymptotic behavior of the orbits, for a Lebesgue-positive set of initial states in M.

We assume that the compact space M is provided with a reference Borel probability measure m, which is given independently of the dynamics. So, m is not necessarily f-invariant, or in other words, we do not restrict the theory to the so called conservative systems. The role of the given reference measure m is to describe how the *initial states* that define the different orbits of the system, are physically chosen.

Precisely, the probability distribution that m provides, states a criteria to measure all the borelian subsets $B \subset M$ of the space, according to the larger or smaller chance in which the initial state drops in B. But notice that m does not usually describe which portions of the space are more or less visited by the future states of the dynamical system, computed on times $n \geq 1$, i.e. after the deterministic dynamics f is acting. It is just a given initial distribution of the points of the space, before the dynamical system f starts its action.

One of the major problems of the modern Ergodic Theory is the existence of the so called physical or SRB (Sinai–Ruelle–Bowen) measures, first studied in [4], [13], [14]. To define them, the given reference probability m in the space M is assumed to be the Lebesgue measure, or equivalent to it in the theoretical measure sense, and after a re-scaling to make $m(M) = 1$. In the sequel, we will denote m to such re-scaled measure, and still call it Lebesgue. We agree to say that a property or conclusion about the dynamics in the future, and in particular about the attractors of the system, is *relevant or observable,* if and only if it holds for all the initial states belonging to a borelian set $B \subset M$ satisfying $m(B) > 0$, namely, for a m-positive probable set of initial conditions. Besides, we say that the property or conclusion is *full probable, or globally observable,* if and only if $m(B) = 1$. Thus, even being usually $B \neq M$ (i.e. B is properly contained in M), if $m(B) = 1$ then a dynamical property $\mathbb{P}$ satisfied by the orbits with initial state in B, is almost always observed. The property $\mathbb{P}$ is full probable in this case. On the contrary, if $m(B) = 0$ then the orbits that satisfy $\mathbb{P}$ come from initial conditions in a set of zero m-probability. In this case the property $\mathbb{P}$ is zero-probable or non observable.

The purpose of the Ergodic Theory is to study the properties of the system in relation with the f- invariant probabilities μ. All the continuous dynamical systems on a compact metric space M do exhibit invariant probabilities (see [10]), and the large majority of such systems exhibit non countably many invariant probabilities. But at the same time, most continuous dynamical systems are not conservative, i.e. the reference Lebesgue measure m according to which the initial state distribute in the space M, is not invariant by f.

Any invariant measure μ describes a spacial distribution of the states, after the system has evolved in time asymptotically to the future, i.e. taking media temporal sequences depending on time n, and then $n \to +\infty$. This latter is the main consequence of the Ergodic Decomposition Theorem (see Section 3 of Chapter 1 of [6]). But only a few dynamical systems, even if one restricts the analysis to the C^1 differentiable dynamical systems, possess relevant invariant measures μ, so called physical or SRB measures. (See for instance [3], [15], [16] to find the open questions about the existence and the properties of the SRB measures.) These latter measures describe the asymptotical spacial distribution of the orbits with their initial states that belong to some m-positive portion of the space. Precisely, if an invariant measure μ is supported in an attractor A whose basin includes a m-positive probable set B, then μ is called a physical measure.

One of the major subjects of research in the modern Differentiable Ergodic Theory, is to find sufficient conditions (if possible generic conditions) of a dynamical system to allow the existence of physical invariant measures. The global conjecture for generic differentiable dynamical systems in [12], the open questions posed in the survey [15], and the state of the art, focused from the Ergodic Theory viewpoint as stated in [16], show the relevance of the problems in this subject. Particularly, for systems that are not sufficiently differentiable, most questions about the existence of SRB or physical measures remain open.

For most C^0 systems (i.e. continuous dynamical systems), even for those that are C^1 (differentiable ones), a single f-invariant measure is not in general enough to describe probabilistically the asymptotic dynamics of some relevant (or observable) portion B of the space M (i.e. satisfying $m(B) > 0$). In fact, in [1] it is announced that generic continuous dynamical systems need infinitely many of its f-invariant measures μ, to describe the asymptotic behavior from initial states in a subset $B \subset M$, such that $m(B) > 0$. As a consequence, there is no hope to find physical or SRB measures, nor ergodic attractors as defined in [11], for generic C^0 dynamical systems. That is why in this paper we revisit the weaker definition of observable weak physical or SRB-like measures that was introduced in [5] (see Definition 3.8). We construct generalized attractors that support those measures, which we call *ergodic-like attractors.*

The first purpose of this paper is to prove that the ergodic-like attractors have the same properties of attraction in mean as the egodic attractors, even if physical or SRB measures do not exist. (Theorems 4.7 and 4.9). The second purpose is to prove that any continuous dynamical system exhibits ergodic-like attractors (Corollary 4.10). To prove those theorems we will revisit the result of existence of observable or weak physical probabilities (Theorem 3.9), taken from [5]. We call all those new results *Ergodic Theorems respect to Lebesgue* because they relate the asymptotic time averages of the orbits of a full Lebesgue set of initial states, with the probability distributions in the ambient manifold M, which are spatial f-invariant measures.

Acknowledgements

The author was partially financed by Comisión Sectorial de Investigación Científica (CSIC) of the Universidad de la República, and Agencia Nacional de Investigación e Innovación (ANII) of Uruguay.

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